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THE COLLINEAR CRACK PROBLEM IN AN INHOMOGENEOUS ORTHOTROPIC MEDIUM

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The Collinear Crack Problem in an Inhomogeneous Orthotropic Medium

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Abstract

The collinear crack problem in an inhomogeneous orthotropic medium is considered under Mode I plane strain or plane stress loading conditions. It is shown that by introducing certain averaged orthotropy parameters, aside from a scaling parameter the results become only weakly dependent on the orthotropy constants. The main results of the study consist of the stress intensity factors at various crack tips as influenced by the material inhomogeneity parameter and by the relative size and position of the secondary cracks with respect to the dominant crack. Also considered is the crack/contact problem for the graded medium subjected to remote tension and bending through fixed grips. It is shown that the crack surface contact on the compression side of the loading has a magnifying effect on the stress intensity factor on the tension side.

1. Introduction

In recent years there has been considerable interest in grading the thermomechanical properties of composites as a new tool in designing materials for specific applications (see Yamanouchi et al. 1990, Holt et al. 1993 and Ilschner and Cherradi 1995 for review, applications and extensive references). Up to now most of the work in the field has been on metal/ceramic particulate composites. By proper selection of the constituent materials, compositional grading and the processing technique, the concept may be used to develop new materials having such highly desirable and seemingly irreconcilable properties as high heat, corrosion and wear resistance, high strength and high toughness in the same material system. The technique may also be used to process interfacial regions to improve bonding (Kurihara et al. 1990) and to reduce the magnitude of residual and thermal stresses in bonded dissimilar materials (Lee and Erdogan 1995, Lee and Erdogan 1997).

In the past the fracture mechanics studies of graded materials were concerned primarily with the influence of material inhomogeneity constants, and certain dimensionless length parameters on the fracture behavior of components containing a single dominant crack. For example, the mode I and the mixed mode crack problems for an infinite isotropic inhomogeneous medium were studied by Delale and Erdogan (1983) and Konda and Erdogan (1994), respectively. The influence of the length parameters on

the stress intensity factors and the strain energy release rates in isotropic graded layers undergoing spallation and surface cracking was investigated by Chen and Erdogan (1994) and Erdogan and Wu (1996), respectively. However, because of the nature of the techniques used in processing, the graded materials are seldom isotropic. Of the two most commonly used processing techniques, generally thermal spray would give a lamellar (Sampath et al. 1995) and the electron beam physical vapor deposition a columnar structure (Kaysser and Ilschner 1995). An appropriate model for such graded materials would be an *orthotropic inhomogeneous* continuum (Ozturk and Erdogan 1997). One additional factor that needs to be considered in studying the basic crack problem in graded materials is the so-called multiple-site cracking or the crack interaction. In homogeneous materials, as intuitively expected, the crack interaction would invariably increase the relative magnitude of stress intensity factors. In graded materials, however, such a categorical statement is, generally not possible and the result may depend on the relative positions of the multiple cracks with respect to the direction of the material property variation as well as on the relative dimensions.

The main objective of this article is to study the influence of the relative size and location of a secondary crack on the stress intensity factors in an orthotropic inhomogeneous medium containing a dominant crack. It is assumed that the cracks are collinear and are located in a plane parallel to the direction of material property variation. It is also assumed that the problem is one of plane strain or generalized plane stress and the solution of the elasticity problem in the absence of crack is known. Thus, the plane of the cracks is one of symmetry and the problem is a mode I crack problem in which the known crack surface tractions are the only external loads. The problem is formulated for arbitrary crack surface tractions and the results are given for some simple loading conditions. The stress intensity factors for more complex loadings may then be obtained by superposition. Clearly, the result would be valid only if the mode I stress intensity factors k_1 at all crack tips are positive. A negative k_1 implies crack closure in which case the problem becomes nonlinear with the size of the contact zone being an additional unknown. This problem is solved by assuming that the graded medium contains a single crack and is subjected to tension and bending away from the crack region through rigid grips (see Appendix A for the solution in a homogeneous isotropic medium which is obtained in closed form).

2. Formulation of the Elasticity Problem

Let E_{11}, E_{22}, G_{12} and $\nu_{ij}, (i,j = 1,2,3)$ be the engineering elastic parameters for an orthotropic inhomogeneous plane x_1, x_2. To replace them we introduce the following four parameters (Krenk 1979, Cinar and Erdogan 1983):

$$E = \sqrt{E_{11}E_{22}}, \quad \nu = \sqrt{\nu_{12}\nu_{21}}, \quad \delta_0^4 = \frac{E_{11}}{E_{22}} = \frac{\nu_{12}}{\nu_{21}}, \qquad \kappa = \frac{E}{2G_{12}} - \nu \tag{1}$$

for plane stress and

$$E = \left(\frac{E_{11}E_{22}}{(1-\nu_{13}\nu_{31})(1-\nu_{23}\nu_{32})}\right)^{1/2}, \nu = \left(\frac{(\nu_{12}+\nu_{13}\,\nu_{32})(\nu_{21}+\nu_{23}\,\nu_{31})}{(1-\nu_{13}\,\nu_{31})(1-\nu_{23}\,\nu_{32})}\right)^{1/2},$$

$$\delta_0^4 = \frac{E_{11}}{E_{22}}\frac{(1-\nu_{23}\nu_{32})}{(1-\nu_{13}\nu_{31})}, \qquad \kappa = \frac{E}{2G_{12}} - \nu \tag{2}$$

for plane strain conditions. In graded materials generally E, ν, κ and δ_0 would be functions of x_1, x_2. However, previously it was shown that the results are only weakly dependent on ν (Delale and Erdogan 1983, Ozturk and Erdogan 1997) and it may be assumed that E_{11}, E_{22} and G_{12} vary proportionately. Consequently, the parameters κ and δ_0 may be assumed to be constant and the material inhomogeneity may be represented by the function $E(x_1, x_2)$ only. Note that $\delta_0 = 1$ and $\kappa = 1$ correspond to the isotropic medium and for $\kappa \leq -1$ the problem has no feasible solution (Erdogan and Wu 1993). By introducing δ_0 as a scaling parameter, in the usual notation we define

$$\begin{aligned}
&x = x_1/\sqrt{\delta_0}, \qquad y = \sqrt{\delta_0}\, x_2,\\
&u(x,y) = u_1(x_1,x_2)\sqrt{\delta_0}, \qquad v(x,y) = u_2(x_1,x_2)/\sqrt{\delta_0},\\
&\sigma_{xx}(x,y) = \sigma_{11}(x_1,x_2)/\delta_0, \qquad \sigma_{yy}(x,y) = \delta_0\,\sigma_{22}(x_1,x_2),\\
&\sigma_{xy}(x,y) = \sigma_{12}(x_1,x_2).
\end{aligned} \tag{3}$$

Furthermore, by assuming that the stiffness E varies in x_1 direction only and in the crack region may be approximated by (Fig. 1)

$$E(x_1,x_2) = E(x_1) = E_0 e^{\alpha x_1} = \widehat{E}(x) = E_0 e^{\gamma x}, \qquad \gamma = \alpha\sqrt{\delta_0}, \tag{4}$$

the equilibrium equations may be expressed as

$$\begin{aligned}
&\frac{\partial^2 u}{\partial y^2} + \beta_1\frac{\partial^2 u}{\partial x^2} + \beta_2\frac{\partial^2 v}{\partial x\partial y} + \beta_1\gamma\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right) = 0,\\
&\frac{\partial^2 v}{\partial x^2} + \beta_1\frac{\partial^2 v}{\partial y^2} + \beta_2\frac{\partial^2 u}{\partial x\partial y} + \gamma\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0,
\end{aligned} \tag{5}$$

where

$$\beta_1 = \frac{2(\kappa+\nu)}{1-\nu^2}, \quad \beta_2 = 1 + \nu\beta_1. \tag{6}$$

Due to symmetry by considering $y > 0$ half plane only, from (5) it may be shown that

$$\begin{aligned}
&u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_3^4 \Lambda_j(k)B_j(k)\,e^{\lambda_j y - ikx}\,dk,\\
&v(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_3^4 B_j(k)\,e^{\lambda_j y - ikx}\,dk, \qquad 0 < y < \infty,
\end{aligned} \tag{7}$$

$$\sigma_{xx}(x,y) = \frac{\widehat{E}(x)}{1-\nu^2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{3}^{4}(\nu\lambda_j - i\Lambda_j k)B_j\, e^{\lambda_j y - ikx}\, dk,$$

$$\sigma_{yy}(x,y) = \frac{\widehat{E}(x)}{1-\nu^2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{3}^{4}(\lambda_j - i\nu\Lambda_j k)B_j\, e^{\lambda_j y - ikx}\, dk,$$

$$\sigma_{xy}(x,y) = \frac{\widehat{E}(x)}{2(\kappa+\nu)}\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{3}^{4}(\lambda_j\Lambda_j - ik)B_j\, e^{\lambda_j y - ikx}\, dk,\ 0 < y < \infty. \tag{8}$$

where

$$\lambda_1 = -\lambda_3 = \left\{\frac{1}{2}(\zeta^2 + 2\kappa\eta) + \frac{1}{2}\sqrt{(\zeta^2+2\kappa\eta)^2 - 4\eta^2}\right\}^{1/2},$$

$$\lambda_2 = -\lambda_4 = \left\{\frac{1}{2}(\zeta^2 + 2\kappa\eta) - \frac{1}{2}\sqrt{(\zeta^2+2\kappa\eta)^2 - 4\eta^2}\right\}^{1/2}, \tag{9}$$

$$\zeta^2 = \nu\gamma^2, \qquad \eta = k^2 + i\gamma k, \tag{10}$$

and

$$\Lambda_j = \frac{(i\beta_2 k - \gamma\beta_1\nu)}{\lambda_j^2 - \beta_1\eta}\lambda_j, \qquad j = 3,4. \tag{11}$$

The unknown functions B_3 and B_4 are determined from the following boundary conditions

$$\sigma_{12}(x_1,0) = 0, \qquad -\infty < x_1 < \infty. \tag{12}$$

$$\sigma_{22}(x_1,+0) = p(x_1), \qquad x_1 \in L, \qquad L = \sum_1^n L_k, \qquad L_k = (a_k, b_k),$$

$$u_2(x_1,+0) = \sqrt{\delta_0}\, v(x,+0) = 0, \qquad x_1 \in L', \qquad L + L' = (-\infty,\infty). \tag{13}$$

Here, it is assumed that the crack surface traction $p(x_1)$ is a known function and the medium contains n collinear cracks along $x_2 = 0$ (Fig. 1).

By defining the following new unknown function

$$\frac{\partial}{\partial x_1}u_2(x_1,+0) = g(x_1), \qquad -\infty < x_1 < \infty, \tag{14}$$

it can be shown that the boundary conditions reduce to (Ozturk and Erdogan 1997)

$$\frac{1}{\pi}\int_L \left[\frac{1}{t_1 - x_1} + R_2(t_1 - x_1)\right] g(t_1)\, dt_1 = \frac{(s_1+s_2)\delta_0}{E_0} e^{-\alpha x_1} p(x_1), \quad x_1 \in L, \tag{15}$$

$$\frac{1}{\pi}\int_{L_j} g(x_1)\,dx_1 = 0, \qquad j = 1,\ldots,n \tag{16}$$

where

$$s_1 = \sqrt{\kappa+\kappa_1}, \qquad s_2 = \sqrt{\kappa-\kappa_1}, \qquad \kappa_1 = \sqrt{\kappa^2-1}, \tag{17}$$

$$R_2(t_1-x_1) = \frac{1}{2\sqrt{\delta_0}}R_1\left((t_1-x_1)/\sqrt{\delta_0}\right)$$

$$R_1(s-x) = \int_0^\infty 2\left[\Re(h(k))\cos(k(s-x)) - \Im(h(k))\sin(k(s-x))\right]dk, \tag{18}$$

$$h(k) = \begin{cases} -i\left[\dfrac{(s_1+s_2)(k+i\gamma)}{\lambda_1+\lambda_2} - 1\right], & \kappa \neq 1 \\ \\ -i\left[\dfrac{2(k+i\gamma)}{\sqrt{\nu\gamma^2+4(k^2+i\gamma k)}} - 1\right], & \kappa = 1. \end{cases} \tag{19}$$

In real materials $-1 < \kappa < \infty$ and $s_1 + s_2$ is always real. Also, for $\gamma = 0$, $\lambda_1 = ks_1$, $\lambda_2 = ks_2$, $h(k) = 0$, $R_2 = 0$ and (15) would reduce to the integral equation for a homogeneous orthotropic medium.

It is interesting to note that in the special case of isotropic inhomogeneous materials the integral in (18) can be evaluated in closed form giving (Appendix B)

$$R_2(t_1-x_1) = \frac{\alpha e^\xi}{2}\left\{\nu_0\frac{|\xi|}{\xi}K_1(\nu_0|\xi|) + K_0(\nu_0|\xi|)\right\} - \frac{1}{t_1-x_1},$$

$$\xi = \frac{\alpha}{2}(t_1-x_1). \tag{20}$$

where K_0 and K_1 are modified Bessel functions. Also note that the kernel R_2 depends on the inhomogeneity constant α and the elastic constants ν and κ but not on E_0 and δ_0. Similarly s_1 and s_2 depend on the shear parameter κ only. It may then be concluded that in (15) E_0 and δ_0 are simply scaling constants and would have no influence on the stress intensity factors which may be defined by and, after solving (15), evaluated from

$$\begin{aligned} k_1(b_k) &= \lim_{x_1\to b_k+0}\sqrt{2(x_1-b_k)}\,\sigma_{22}(x_1,+0) \\ &= -\lim_{x_1\to b_k-0}\frac{E(x_1)}{(s_1+s_2)\delta_0}\sqrt{2(b_k-x_1)}\,g(x_1), \end{aligned}$$

$$\begin{aligned} k_1(a_k) &= \lim_{x_1\to a_k-0}\sqrt{2(a_k-x_1)}\,\sigma_{22}(x_1,+0) \\ &= \lim_{x_1\to a_k+0}\frac{E(x_1)}{(s_1+s_2)\delta_0}\sqrt{2(x_1-a_k)}\,g(x_1),\ k = 1,\cdots,n. \end{aligned} \tag{21}$$

Also, from (13) and (14) the crack surface displacements may be obtained as

$$u_2(x_1,+0) = \int_{a_j}^{x_1} g(t_1)dt_1, \qquad a_j < x_1 < b_j, \qquad j = 1,\ldots,n. \tag{22}$$

3. On The Solution Of The Integral Equation

Since the closed form solution is not available the integral equation (15) must be solved numerically. A simple and very effective way to do this would be to reduce (15) to a system of n integral equations each having the support $(-1,\,1)$ by defining

$$q = \frac{2t_1 - (b_k + a_k)}{(b_k - a_k)}, \qquad a_k < t_1 < b_k, \qquad -1 < q < 1,$$

$$r = \frac{2x_1 - (b_k + a_k)}{(b_k - a_k)}, \qquad a_k < x_1 < b_k, \qquad -1 < r < 1,$$

$$\phi_k(q) = g(t_1), \qquad a_k < t_1 < b_k, \qquad -1 < q < 1,$$

$$f_k(r) = p(x_1)/P_0, \qquad a_k < x_1 < b_k, \qquad -1 < r < 1,$$

$$R_{ik}(c_k(q - z_{ik})) = R_2(t_1 - x_1)/c_k, \quad a_i < x_1 < b_i, \quad a_k < t_1 < b_k,$$
$$-1 < q < 1,$$

$$c_k = \frac{1}{2}(b_k - a_k), \quad d_k = \frac{1}{2}(b_k + a_k),$$

$$z_{ik} = \frac{c_i}{c_k} r + \frac{1}{c_k}(d_i - d_k), \quad (i,k) = 1,\cdots,n, \quad -1 < r < 1, \tag{23}$$

where P_0 which has the dimension of stress is a normalization constant. The equations (15) and (16) may then be expressed as

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{q-r} + c_i R_{ii}\right]\phi_i(q)\,dq$$
$$+\sum_{\substack{k=1\\k\neq i}}^{n}\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{q-z_{ik}} + c_k R_{ik}\right]\phi_k(q)dq = \frac{(s_1+s_2)\delta_0 P_0}{E_0\exp(\alpha(c_i r + d_i))} f_i(r),$$
$$-1 < r < 1, \quad i = 1,\cdots,n, \tag{24}$$

$$\int_{-1}^{1}\phi_k(q)\,dq = 0, \qquad k = 1,\cdots,n. \tag{25}$$

The system of integral equations may be solved in a standard manner by expressing

$$\phi_k(q) = \frac{(s_1+s_2)\delta_0 P_0}{E_0 \exp(\alpha d_k)} \frac{1}{\sqrt{1-q^2}} \sum_{n=0}^{\infty} A_{km} T_m(q), \quad -1 < q < 1, \quad k = 1, \cdots, n, \tag{26}$$

where $T_m(q)$ is the Chebyshev polynomial of the first kind and A_{km} are unknown dimensionless constants. From (25) and othogonality conditions of T_m's it follows that $A_{k0} = 0$. The coefficients A_{km} are determined by substituting from (26) into (24) and by using the method of reduction (Kantorovich and Krylov 1958, Erdogan 1978). The stress intensity factors may then be obtained from (21) and (26) as follows:

$$k_1(b_k) = -P_0\sqrt{c_k}\, e^{\alpha c_k} \sum_{m=1}^{\infty} A_{km},$$

$$k_1(a_k) = P_0\sqrt{c_k}\, e^{-\alpha c_k} \sum_{m=1}^{\infty} (-1)^m A_{km}, \quad k = 1, \cdots, n. \tag{27a,b}$$

4. Results and Discussion

The main interest in this study is in evaluating the influence of relative position, size and distance of a "small" crack lying in the plane of a dominant crack on the stress intensity factors in an inhomogeneous orthotropic medium. With the additional knowledge of subcritical crack growth characteristics of the material, one may then be able to determine, for example, fatigue or corrosion crack growth rate at each crack tip. The problem is solved under two sets of loading conditions. In the first, it is assumed that the crack surface tractions are given by (see 13)

$$p(x_1) = -p_0 - p_1\left(\frac{x_1}{a}\right) - p_2\left(\frac{x_1}{a}\right)^2 - p_3\left(\frac{x_1}{a}\right)^3, \quad x_1 \in L, \tag{28}$$

where p_0, p_1, p_2 and p_3 are the measure of the magnitude of external loads and are known constants and a is a normalizing length parameter (usually the half length of the dominant crack). Since the problem is linear, these results may be helpful to find an approximate solution to the crack problem for a given specific loading. The second external load considered is a remote displacement loading of the form

$$\epsilon_{22}(x_1, x_2) = \epsilon_0 + \epsilon_1(x_1/a) \tag{29}$$

giving

$$p(x_1) = -\frac{E_0}{\delta_0^2} e^{\alpha x_1}\left(\epsilon_0 + \epsilon_1\left(\frac{x_1}{a}\right)\right), \quad x_1 \in L. \tag{30}$$

Summary of the results for an isotropic inhomogeneous medium containing a single crack along $x_2 = 0$, $-a < x_1 < a$ is shown in Tables 1-3. These results are both accurate and comprehensive. Tables 4-6 show the comparison of the stress intensity factors obtained in this study with that found by Delale and Erdogan (1983). From the solution given in the previous section the crack surface displacement may easily be expressed as

$$\frac{u_2(x_1,+0)}{v_0} = -\sqrt{1-\left(\frac{x_1}{a}\right)^2}\sum_1^{\infty}\frac{A_n}{n}U_{n-1}\left(\frac{x_1}{a}\right), \tag{31}$$

where $u_2(x_1,+0) = v(x,0)$,

$$v_0 = 2aP_0(1-\nu^2)/\widehat{E}_0 \tag{32}$$

for isotropic and

$$v_0 = (s_1+s_2)\delta_0 P_0 a/\widehat{E}_0 \tag{33}$$

for orthotropic inhomogeneous materials ($\widehat{E}_0 = E_0$ for plane stress and $\widehat{E}_0 = E_0/(1-\nu^2)$ for plane strain). Some sample results for the crack surface displacement are shown in Figure 2.

The calculated plane strain results for the normalized stress intensity factors at various crack tips in an isotropic inhomogeneous medium containing two unequal cracks under fixed grip ($\varepsilon_{22} = \varepsilon_0$) or fixed load ($\sigma_{yy}(x_1,0) = -p_0$) conditions are given in Figures 3-16. Some of the limiting cases of these results for $(c_1+c_2)/d \to 0$ (the uncoupled case) and $(c_1+c_2)/d \to 1$ (the case of a single crack of length $2(c_1+c_2)$) may be found in Table 7. One may note that as $(c_1+c_2)/d \to 1$ or $a_2 \to b_1$, $k_1(b_1) \to \infty$, $k_1(a_2) \to \infty$ and $k_1(a_1)$ and $k_1(b_2)$ approach the single crack values. As shown by Boduroglu and Erdogan (1983) and as may be observed in Figure 7, for $a_2 \to b_1$ $k_1(a_1)$ and $k_1(b_2)$ are highly ill-defined (that is, their slopes are unbounded as $(c_1+c_2)/d \to 1$).

Some sample results for stress intensity factors in an orthotropic inhomogeneous medium containing two unequal collinear cracks under plane strain conditions are given in Figures 17-21 and Tables 8 and 9. Aside from the geometric parameters c_2/c_1 and $(c_1+c_2)/d$, as pointed out previously, the main variables in this case are the material inhomogeneity constant α and the shear parameter κ. Both the figures and the tables show that the dependence of the stress intensity factors on α is strong and on κ is rather weak.

The results for the crack/contact problem under remote loading $\varepsilon_{22} = \varepsilon_0 + \varepsilon_1(x/a)$ are shown in Tables 10, 11 and Figures 22-25. Figure 22 shows the crack surface displacement for $\varepsilon_0/\varepsilon_1 = \beta = 0$ (remote bending) obtained by ignoring the crack surface interpenetration. Thus, on the compressive side of the external load the solution gives negative displacement which is physically unacceptable. Figures 23 and 24 show the crack surface displacement for $\beta = 0$ obtained by taking into account the smooth contact of the surfaces. Note the slight dependence of the crack surface displacement on the sign of ε_1. Table 10 shows the stress intensity factors and the size of the contact zone again for remote loading $\varepsilon_{22} = \varepsilon_1(x_1/a)$ and for various values of the material inhomogeneity constant α, $(E(x_1) = E_0\exp(\alpha x_1)$, see (4)). The contact occurs along $x_2 = 0$, $-a < x_1 < -b$, $(-a < -b)$ for $\varepsilon_1 > 0$ and along $x_2 = 0$, $b < x_1 < a$, $(b < a)$ for $\varepsilon_1 < 0$. The table also shows the (theoretical) stress intensity factors calculated by disregarding the crack surface penetration (no contact case). Note that the presence of contact tends to magnify the stress intensity factor at the crack tip on the tension side (

$k_1(a)$ for $\varepsilon_1 > 0$ and $k_1(-a)$ for $\varepsilon_1 < 0$). Note also the highly significant influence of the material inhomogeneity constant on the stress intensity factors.

Some sample results for the combined loading "tension" and "bending", $\varepsilon_{22} = \varepsilon_0 + \varepsilon_1(x_1/a)$, $(\varepsilon_0/\varepsilon_1 = \beta)$ are shown in Figure 25 and Table 11. Figure 25 shows the crack surface displacement for various values of β. For $\beta \geq \beta_c$ and the contact zone size becomes zero, that is $b = a$, $\beta = \beta_c$ corresponding to $k_1(-a) = 0$. Also, $b < a$ for $\beta < \beta_c$. Table 11 shows the crack contact zone sizes and the corresponding stress intensity factors. If the material is homogeneous and isotropic the crack/contact problem can be solved in closed form which is given in Appendix A.

Finally, a sample result giving the distribution of $\sigma_{yy}(x, 0)$ along the net ligament $b_1 < x < a_2$ is shown in Fig. 26. Note that the solution given in this study is that of the perturbation problem which must be added to the results obtained from the uncracked medium in order to find the total solution.

5. Conclusions

In the case of two collinear cracks with crack tips at (a_1, b_1) and (a_2, b_2) and $a_1 < b_1 < a_2 < b_2$, $b_1 - a_1 \gg b_2 - a_2$ (that is, (a_1, b_1) being the dominant crack), if the medium is homogeneous (isotropic or orthotropic) then under mode I loading generally $k_1(b_1) > k_1(a_1) > k_1(a_2) > k_1(b_2)$, meaning that fastest crack growth would take place at the inner tip of the dominant crack. As the ligament $a_2 - b_1$ decreases and tends to zero $k_1(a_2)$ as well as $k_1(b_1)$ becomes unbounded. However, if the medium is inhomogeneous, the values of the stress intensity factors may be influenced quite significantly by the relative positions of the dominant and the secondary cracks with respect to the direction of material property variation.

After introducing the "averaged" orthotropy parameters E, ν, κ, and δ_0, in mode I crack problems for an inhomogeneous orthotropic medium $E(0)$ and δ_0 act as scaling constants and ν and κ do not seem to have a significant influence on the stress intensity factors. In crack problems involving graded materials the dominant factor remains to be the material inhomogeneity constant.

In mode I and mode III crack problems for isotropic graded materials described by $\mu(x) = \mu_0 \exp(\alpha x)$ the kernels of the integral equations can be evaluated in closed form. Consequently, for these materials highly accurate benchmark solutions can be obtained. In this article the mode I stress intensity factors in such materials are provided for tension and bending through fixed grips as well as third degree polynomial tractions acting on the crack surfaces.

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APPENDIX A

Crack Closure for a Homogeneous Isotropic Medium

Let the medium contain a single crack of length $2a$ along the x axis and be subjected to crack surface tractions $\sigma_{xy}(x,0) = 0$, $\sigma_{yy}(x,0) = p(x)$, $-a < x < a$. By defining $\partial v(x,+0)/\partial x = g(x)$, the integral equation of the problem may be expressed as

$$\frac{1}{\pi}\int_{-a}^{a} \frac{g(s)\,ds}{s-x} = \frac{2}{E_0}p(x), \quad -a < x < a, \tag{A1}$$

subject to

$$\frac{1}{\pi}\int_{-a}^{a} g(s)\,ds = 0, \tag{A2}$$

where $E_0 = E$ for plane stress and $E_0 = E/(1-\nu^2)$ for plane strain conditions. By substituting

$$x = ax', \quad s = as', \quad g(s) = G(s'), \quad p(x) = P(x'). \tag{A3}$$

A1 and A2 become

$$\frac{1}{\pi}\int_{-1}^{1} \frac{G(s')\,ds'}{s'-x'} = \frac{2}{E_0}P(x'), \quad \int_{-1}^{1} G(s')\,ds' = 0. \tag{A4}$$

We now assume that the solution of (A4) is of the form

$$G(s') = \sum_{0}^{\infty} \frac{A_n T_n(s')}{\sqrt{1-s'^2}}. \tag{A5}$$

Considering the orthogonality condition

$$\frac{1}{\pi}\int_{-1}^{1} \frac{T_n(q)T_m(q)}{\sqrt{1-q^2}}dq = \begin{cases} 1, & m = n = 0 \\ 1/2, & m = n \geq 1, \\ 0, & m \neq n, \end{cases} \tag{A6}$$

from (A4b) it may be seen that $A_0 = 0$. Also, by substituting from (A5) into (A4a) and by using the properties

$$\frac{1}{\pi}\int_{-1}^{1} \frac{T_n(q)}{(q-r)\sqrt{1-q^2}}dq = \begin{cases} 0, & n = 0, \quad |r| < 1 \\ U_{n-1}(r), & n \geq 1, \quad |r| < 1, \\ G_n(r), & n = 0, 1, \ldots, \quad |r| > 1, \end{cases} \tag{A7}$$

$$T_n(q) = \cos(n\theta),\, U_n(q) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad \cos\theta = q, \tag{A8}$$

$$G_n(r) = -\frac{|r|}{r}\frac{\left(r - (|r|/r)\sqrt{r^2-1}\right)^n}{\sqrt{r^2-1}}, \tag{A9}$$

we obtain

$$\sum_{m=1}^{\infty} A_m U_{m-1}(x') = \frac{2}{E_0}P(x'). \tag{A10}$$

As an example consider now the remote "bending" of the medium for which we have

$$P(x') = -E_0\varepsilon_1 x' = -E_0\varepsilon_1 U_1(x')/2. \tag{A11}$$

From (A10) it then follows that

$$A_1 = 0, \quad A_2 = -\varepsilon_1, \quad A_m = 0, \quad m \geq 3. \tag{A12}$$

and the solution becomes

$$G(s') = -\frac{\varepsilon_1 T_2(s')}{\sqrt{1-s'^2}} = -\frac{\varepsilon_1(2s'^2-1)}{\sqrt{1-s'^2}}. \tag{A13}$$

giving the stress intensity factors

$$k_1(a) = \frac{1}{2}E_0\varepsilon_1\sqrt{a}, \quad k_1(-a) = -\frac{1}{2}E_0\varepsilon_1\sqrt{a}. \tag{A14}$$

Since the negative stress intensity factor (A14b) implies interpenetration of the crack surfaces, the solution given by (A13) is not valid. The correct solution may be obtained by taking into account the crack surface contact near the end $x = -a$. Let the contact region be $-a < x < -b$ with b being an unknown constant to be determined from the smoothness condition $k_1(-b) = 0$. The integral equation (A1) and the single-valuedness condition (A2) may then be modified as

$$\frac{1}{\pi}\int_{-b}^{a}\frac{g(s)\,ds}{s-x} = \frac{2}{E_0}p(x), \quad -b < x < a, \tag{A15}$$

$$\frac{1}{\pi}\int_{-b}^{a} g(s)\,ds = 0, \tag{A16}$$

By defining the following normalizations

$$s = cS + d, x = cX + d, c = \frac{a+b}{2}, d = \frac{a-b}{2}, g(s) = G(S),\, p(x) = P(X), \tag{A17}$$

(A15) becomes

$$\frac{1}{\pi}\int_{-1}^{1}\frac{G(S)\,dS}{S-X}=\frac{2}{E_0}P(X), \quad -1<X<1. \tag{A18}$$

Applying again remote "bending" we have

$$p(x)=-E_0\varepsilon_1\left(\frac{x}{a}\right)=-E_0\varepsilon_1\left[\frac{c}{a}X+\frac{d}{a}\right]=P(X) \tag{A19}$$

The integral equation (A18) then becomes

$$\frac{1}{\pi}\int_{-1}^{1}\frac{G(S)\,dS}{S-X}=-2\varepsilon_1\left[\frac{c}{a}X+\frac{d}{a}\right], \quad -1<X<1, \tag{A20}$$

Again assuming the solution as given by (A5) it can be shown that

$$\sum_{m=1}^{\infty}A_m U_{m-1}(X)=-2\varepsilon_1\left[\frac{c}{a}X+\frac{d}{a}\right]. \tag{A21}$$

By observing that $U_0(X)=1$ and $U_1(X)=2X$, from (A21) it follows that

$$A_1=-2\varepsilon_1\frac{d}{a}, \quad A_2=-\varepsilon_1\frac{c}{a}, \quad A_m=0, \quad m>2. \tag{A22}$$

Thus, the solution becomes

$$G(S)=-\frac{\varepsilon_1}{a}\frac{1}{\sqrt{1-S^2}}\left[2dS+c\left(1-2S^2\right)\right], \tag{A23}$$

giving

$$k_1(a)=E_0\varepsilon_1\sqrt{c}\left(\frac{3a-b}{4a}\right), \tag{A24}$$

$$k_1(-b)=-E_0\varepsilon_1\sqrt{c}\left(\frac{3b-a}{4a}\right). \tag{A25}$$

From the smooth contact condition $k_1(-b)=0$, the unknown constant b is found to be

$$b=\frac{1}{3}a. \tag{A26}$$

From A24 and A26 it then follows that

$$\frac{k_1(a)}{E_0\varepsilon_1\sqrt{c}}=\frac{2}{3} \text{ or } \frac{k_1(a)}{E_0\varepsilon_1\sqrt{a}}=\frac{2}{3}\sqrt{2/3}=0.5443. \tag{A27}$$

Figure A1 shows the crack surface displacements obtained from (A13) by ignoring the interpenetration of crack surfaces and from (A23) by taking into account crack closure

along $-a < x < -b$. From Figure A1 and equation (A27) it is seen that the crack closure near the end $x = -a$ leads to magnification of stress state near the crack tip $x = a$ and there is approximately a 9% increase in the stress intensity factor $k_1(a)$ over the nominal value obtained by ignoring crack surface contact.

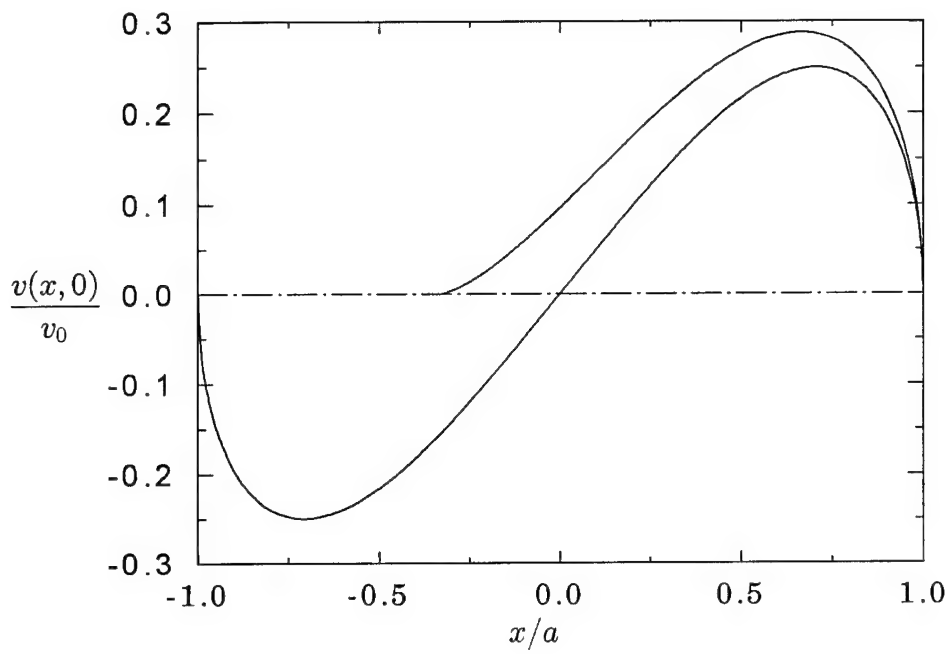

Figure A1 Crack surface displacements in a homogeneous isotropic medium subjected to remote bending $\varepsilon_{yy} = \varepsilon_1(x/a)$ and obtained by ignoring crack closure and by taking it into account; $v_0 = 2\varepsilon_1 a$.

After solving the perturbation problem (A15) the stress $\sigma_{yy}(x,0)$ on the plane of the crack may be obtained by using (A7), (A9), (A11) and (A18) and by adding the homogeneous stress $\sigma_{yy} = E_0\varepsilon_1(x/a)$ as follows:

$$\sigma_{yy}(x,0) = \begin{cases} 0, \quad -b < x < a, \\ \dfrac{E_0}{2}[A_1G_1(X) + A_2G_2(X)] + E_0\varepsilon_1\left(\dfrac{c}{a}X + \dfrac{d}{a}\right), \end{cases}$$

$$X = \frac{x-d}{c}, \quad c = \frac{a+b}{2}, \quad d = \frac{a-b}{2}, \quad -\infty < x < -b, \quad a < x < \infty, \qquad \text{A28}$$

or by observing that $b = a/3$ it may be shown that

$$\frac{\sigma_{yy}(x,0)}{E_0\varepsilon_1} = \begin{cases} 0, \quad |X| < 1, \\ \dfrac{|X|}{3X}\dfrac{1}{\sqrt{X^2-1}}\left[X - \dfrac{|X|}{X}\sqrt{X^2-1} + \left(X - \dfrac{|X|}{X}\sqrt{X^2-1}\right)^2\right] \\ \qquad + \dfrac{2X+1}{3}, \quad |X| > 1. \end{cases} \qquad \text{A29}$$

Figure A2 shows the stress distribution expressed by (A29). Note that, as expected, $\sigma_{yy}(x,0) < 0$ for $x < -b$.

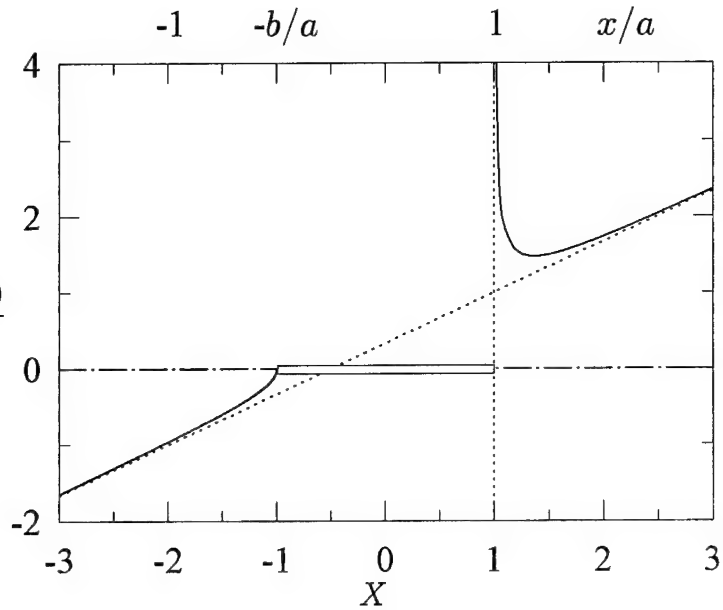

Figure A2 The stress σ_{yy} in the plane of the crack in a homogeneous isotropic medium under remote "bending" $\varepsilon_{yy} = \varepsilon_1(x/a)$.

Let us now assume that a homogeneous isotropic medium containing a crack along $-a < x < a$, $y = 0$ is subjected to remote "tension" $\varepsilon_{yy} = \varepsilon_0$ as well as "bending" $\varepsilon_{yy} = \varepsilon_1(x/a)$. The input function in the integral equation (A1) would then be

$$p(x) = -E_0\left[\varepsilon_0 + \varepsilon_1\left(\frac{x}{a}\right)\right] = -E_0\varepsilon_1\left[\beta + \left(\frac{x}{a}\right)\right], \quad \beta = \frac{\varepsilon_0}{\varepsilon_1}, \qquad \text{A30}$$

giving the solution as follows:

$$G(s') = \frac{1}{\sqrt{1-s'^2}}[A_1T_1(s') + A_2T_2(s')], \quad s' = s/a, \qquad \text{A31}$$

$$A_1 = -2\varepsilon_1\beta, \quad A_2 = -\varepsilon_1, \qquad \text{A32}$$

$$\frac{k_1(a)}{E_0\varepsilon_1\sqrt{a}} = \frac{1}{2}(2\beta+1), \qquad \text{A33}$$

$$\frac{k_1(-a)}{E_0\varepsilon_1\sqrt{a}} = \frac{1}{2}(2\beta-1), \qquad \text{A34}$$

$$\frac{v(x,0)}{v_0} = \left(\beta + \frac{x}{2a}\right)\sqrt{1-(x/a)^2}, \qquad -a < x < a. \qquad \text{A35}$$

The solution given by (31)-(35) is valid provided $k_1(-a) > 0$. From (A34) the critical value of the strain ratio $\beta = \varepsilon_0/\varepsilon_1$ for which $b = a$ or $k_1(-a) = 0$ is found to be

$$(\varepsilon_0/\varepsilon_1)_c = \beta_c = 1/2. \qquad \text{A36}$$

Thus, crack closure would take place if $\beta < \beta_c$ (or if $k_1(-a) < 0$). In this case the solution is obtained from (A15) and (A30) by assuming that $b < a$ and by following the procedure outlined by (A17)-(A27). It may then be shown that

$$G(S) = \frac{1}{\sqrt{1-S^2}}[A_1T_1(S) + A_2T_2(S)], \quad S = (s-d)/c, \qquad \text{A37}$$

$$A_1 = -2\varepsilon_1\left(\beta + \frac{d}{a}\right), \; A_2 = -\varepsilon_1\left(\frac{c}{a}\right), \qquad \text{A38}$$

$$\frac{b}{a} = \frac{1}{3}(1+4\beta), \qquad \frac{c}{a} = \frac{2}{3}(1+\beta), \qquad \frac{d}{a} = \frac{1}{3}(1-2\beta), \qquad \text{A39}$$

$$\frac{k_1(a)}{E_0\varepsilon_1\sqrt{a}} = \left(\frac{2}{3}(1+\beta)\right)^{3/2}, \qquad \text{A40}$$

$$\frac{k_1(-b)}{E_0\varepsilon_1\sqrt{a}} = 0, \qquad \text{A41}$$

$$\frac{v(x,0)}{v_0} = \frac{2}{9}(1+\beta)^2(1+X)\sqrt{1-X^2}, \quad v_0 = 2\varepsilon_1 a, \quad x = cX + d, \quad |X| < 1. \qquad \text{A42}$$

Figure A3 shows the crack surface displacement $v(x,0)$, $-b < x < a$, the size of the contact zone b and the normalized stress intensity factors for various values of the strain ratio $\beta = \varepsilon_0/\varepsilon_1$.

$\beta = \dfrac{\varepsilon_0}{\varepsilon_1}$	b/a	$\dfrac{k_1(-a)}{E_0\varepsilon_1\sqrt{a}}$	$\dfrac{k_1(a)}{E_0\varepsilon_1\sqrt{a}}$
1.0	1	0.5	1.5
0.75	1	0.25	1.25
0.5	1	0.0	1.0
0.25	2/3	0.0	0.7607
0.0	1/3	0.0	0.5443

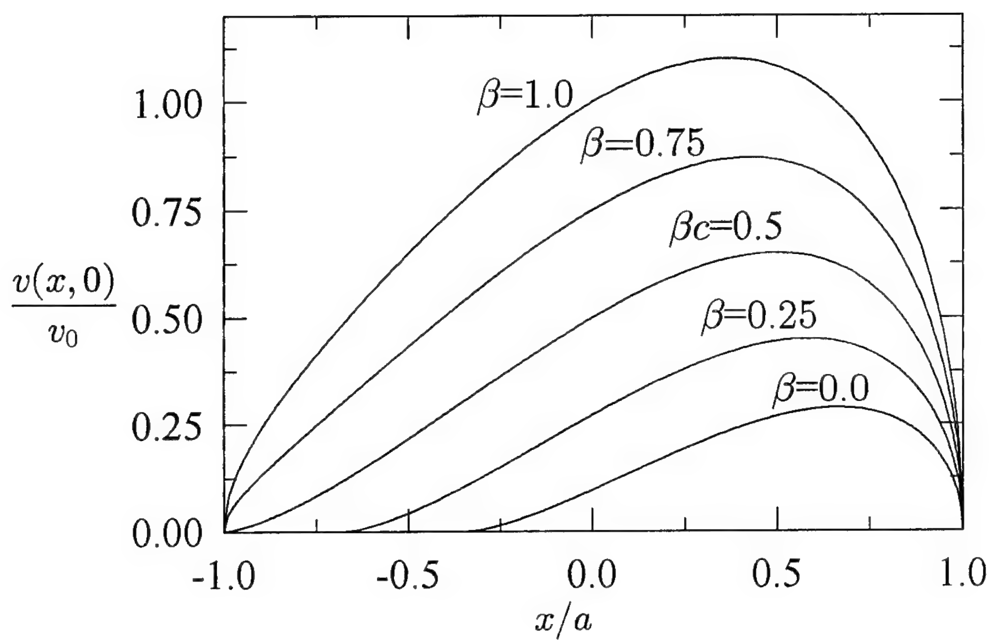

Figure A3 The crack surface displacement, the contact zone size and the normalized stress intensity factors in a cracked isotropic homogeneous medium subjected to the remote loading $\varepsilon_{yy} = \varepsilon_0 + \varepsilon_1(x/a),\ v_0 = 2\varepsilon_1 a$.

APPENDIX B

Evaluation of the Kernel

For an *isotropic* inhomogeneous medium containing a series of collinear cracks along the x axis the kernel of the integral equation (15) may be expressed as

$$R(\omega) = \frac{2}{i}\int_{-\infty}^{\infty} \frac{k + i\alpha}{\sqrt{\nu\alpha^2 + 4(k^2 + i\alpha k)}}\, \mathrm{e}^{ik\omega} dk = 2\left(\frac{1}{\omega} + R_2(\omega)\right), \quad \omega = s - x. \qquad \text{B1}$$

Defining $A = \alpha/2$ and observing that $k^2 + i\alpha k = (k + iA)^2 + A^2$, from (B1) it follows that

$$R(\omega) = \frac{1}{i}\int_{-\infty}^{\infty} \frac{(k + i2A)}{\sqrt{(\nu + 1)A^2 + (k + iA)^2}}\, \mathrm{e}^{ik\omega} dk. \qquad \text{B2}$$

Changing the variables $k = AV$ and assuming $A > 0$, (B2) becomes

$$R(\omega) = \frac{A}{i}\int_{-\infty}^{\infty} \frac{(V + 2i)}{\sqrt{(\nu + 1) + (V + i)^2}}\, \mathrm{e}^{iAV\omega} dV, \qquad \text{B3}$$

or, by letting $U = V + i$, $\nu + 1 = C^2$ and $\mathcal{B} = A\omega$ we find

$$R(\omega) = \frac{A\mathrm{e}^{\mathcal{B}}}{i}\int_{-\infty+i}^{\infty+i} \frac{U + i}{\sqrt{C^2 + U^2}}\, \mathrm{e}^{i\mathcal{B}U} dU. \qquad \text{B4}$$

By substituting $U = -U'$ and observing that

$$\int_{\infty-i}^{-i} \frac{(-U' + i)}{\sqrt{C^2 + U'^2}} \mathrm{e}^{-i\mathcal{B}U'}(-dU') = -\int_{-i}^{\infty-i} \frac{(U' - i)}{\sqrt{C^2 + U'^2}} \mathrm{e}^{-i\mathcal{B}U'} dU', \qquad \text{B5}$$

(B4) may be written as

$$\begin{aligned} R(\omega) = A\mathrm{e}^{\mathcal{B}}\Bigg\{&\int_{-i}^{\infty-i} \left(-\frac{1}{i}\right) \frac{(U - i)}{\sqrt{C^2 + U^2}}\, \mathrm{e}^{-i\mathcal{B}U} dU \\ &+ \int_{i}^{\infty+i} \left(\frac{1}{i}\right) \frac{(U + i)}{\sqrt{C^2 + U^2}}\, \mathrm{e}^{i\mathcal{B}U} dU\Bigg\} \\ = 2A\mathrm{e}^{\mathcal{B}}\Re&\left\{\int_{i}^{\infty+i} \frac{(U + i)}{i\sqrt{C^2 + U^2}}\, \mathrm{e}^{i\mathcal{B}U} dU\right\}. \end{aligned} \qquad \text{B6}$$

We now consider

$$I_1 = \int_i^{\infty+i} \frac{(U+i)}{i\sqrt{C^2+U^2}} \mathrm{e}^{i\mathcal{B}U} dU, \quad \mathcal{B} > 0. \qquad \text{B7}$$

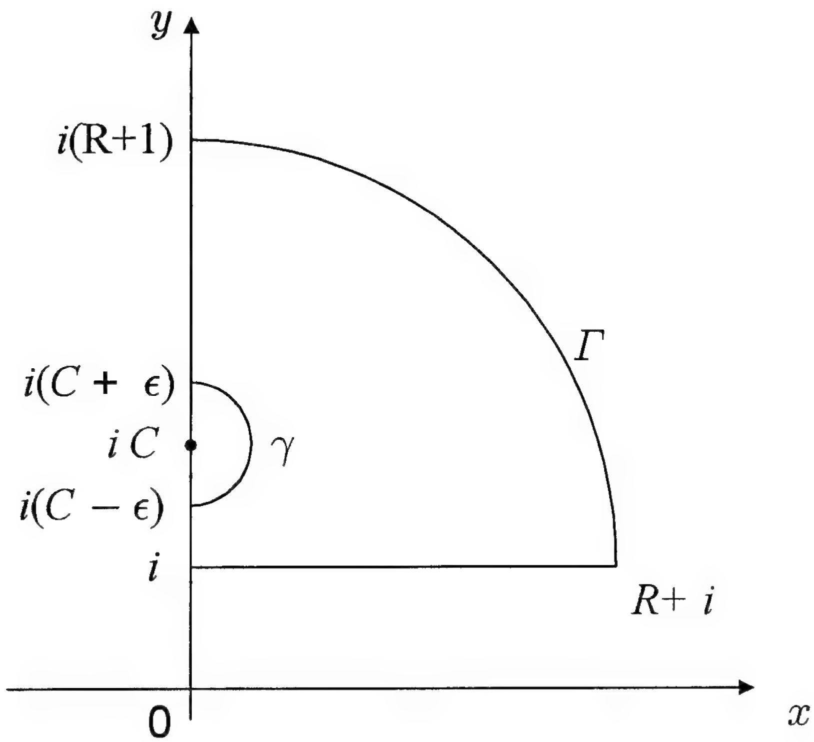

Figure B1 Contour used in evaluating I_1

Referring to Figure B1 we have

$$\oint \frac{(z+i)}{i\sqrt{z^2+C^2}} \mathrm{e}^{i\mathcal{B}z} dz = \int_i^{R+i} + \int_\Gamma + \int_{i(R+1)}^{i(C+\varepsilon)} + \int_\gamma + \int_{i(C-\varepsilon)}^{i} = 0, \qquad \text{B8}$$

For $R \to \infty$ and $\varepsilon \to 0$ it can be shown that the integrals over the arcs γ and Γ in (B8) are zero and from (B8) it follows that

$$I_1 = \int_i^{\infty+i} \frac{(U+i)}{i\sqrt{C^2+U^2}} \mathrm{e}^{i\mathcal{B}U} dU = \int_C^{\infty} \frac{y+1}{\sqrt{y^2-C^2}} \mathrm{e}^{-\mathcal{B}y}\, dy$$

$$+ i \int_1^C \frac{y+1}{\sqrt{C^2-y^2}} \mathrm{e}^{-\mathcal{B}y}\, dy. \qquad \text{B9}$$

By substituting $y = CY$ in the first integral, (B9) becomes

$$I_1 = \int_i^{\infty+i} \frac{(U+i)}{i\sqrt{C^2+U^2}}\,\mathrm{e}^{i\mathcal{B}U}\,dU = \int_1^{\infty} \frac{CY+1}{\sqrt{Y^2-1}}\mathrm{e}^{-\mathcal{B}CY}\,dY$$

$$+\,i\int_1^{C} \frac{y+1}{\sqrt{C^2-y^2}}\mathrm{e}^{-\mathcal{B}y}\,dy. \qquad \text{B10}$$

From (B6), (B7) and (B10) it then follows that

$$R(\omega) = 2A\mathrm{e}^{\mathcal{B}}\Re\left\{\int_i^{\infty+i} \frac{(U+i)}{i\sqrt{C^2+U^2}}\,\mathrm{e}^{i\mathcal{B}U}\,dU\right\} = 2A\mathrm{e}^{\mathcal{B}}\int_1^{\infty} \frac{CY+1}{\sqrt{Y^2-1}}\mathrm{e}^{-\mathcal{B}CY}\,dY \qquad \text{B11}$$

By using the following expressions of the modified Bessel functions

$$K_0(\rho) = \int_1^{\infty} \frac{1}{\sqrt{x^2-1}}\mathrm{e}^{-\rho x}\,dx, \quad K_1(\rho) = \int_1^{\infty} \frac{x}{\sqrt{x^2-1}}\mathrm{e}^{-\rho x}\,dx, \qquad \text{B12}$$

(B11) becomes

$$R(\omega) = 2A\mathrm{e}^{\mathcal{B}}\{CK_1(\mathcal{B}C) + K_0(\mathcal{B}C)\} = 2\left(\frac{1}{\omega} + R_2(\omega)\right). \qquad \text{B13}$$

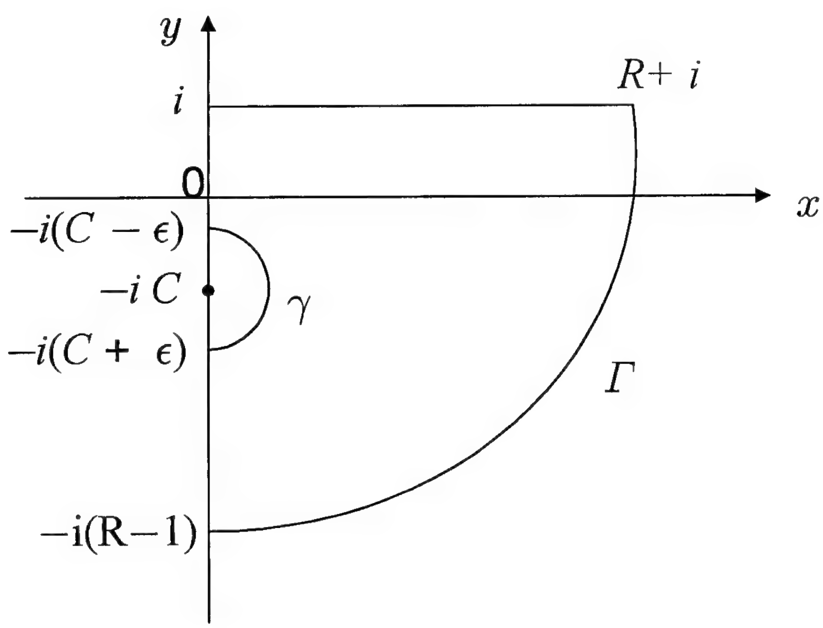

Figure B2 Contour used in evaluating I_2

Let us now assume that $\mathcal{B} < 0$, $\mathcal{B} = -\mathcal{B}'$ and define the function

$$I_2 = \int_i^{\infty+i} \frac{(U+i)}{i\sqrt{C^2+U^2}}\, \mathrm{e}^{-i\mathcal{B}'U} dU, \quad \mathcal{B}' > 0, \tag{B14}$$

Referring to Figure B2 we have

$$\oint \frac{(z+i)}{i\sqrt{C^2+z^2}}\, \mathrm{e}^{-i\mathcal{B}'z} dz = \int_i^{R+i} + \int_\Gamma + \int_{-i(R-1)}^{-i(C+\varepsilon)} + \int_\gamma + \int_{-i(C-\varepsilon)}^{i} = 0, \tag{B15}$$

Again, for $R \to \infty$ and $\varepsilon \to 0$ it can be shown that the integrals over the arcs γ and Γ in B15 are zero and I_2 becomes

$$\begin{aligned} I_2 = \int_i^{\infty+i} \frac{(U+i)}{i\sqrt{C^2+U^2}}\, \mathrm{e}^{-i\mathcal{B}'U} dU &= \int_{-iC}^{-i\infty} \frac{(z+i)}{i\sqrt{C^2+z^2}}\, \mathrm{e}^{-i\mathcal{B}'z} dz \\ &\quad - i\int_{-iC}^{i} \frac{(z+i)}{i\sqrt{C^2+z^2}}\, \mathrm{e}^{-i\mathcal{B}'z} dz \\ &= \int_C^\infty \frac{(-y+1)}{\sqrt{y^2-C^2}}\, \mathrm{e}^{-\mathcal{B}'y} dy \; - i\int_{-C}^{1} \frac{(y+1)}{\sqrt{C^2-y^2}}\, \mathrm{e}^{-\mathcal{B}'y} dy, \end{aligned} \tag{B16}$$

or

$$\begin{aligned} \Re\left\{\int_i^{\infty+i} \frac{(U+i)}{i\sqrt{C^2+U^2}}\, \mathrm{e}^{-i\mathcal{B}'U} dU\right\} &= \int_C^\infty \frac{(-y+1)}{\sqrt{y^2-C^2}}\, \mathrm{e}^{-\mathcal{B}'y} dy \\ &= \int_1^\infty \frac{(-CY+1)}{\sqrt{Y^2-1}}\, \mathrm{e}^{-\mathcal{B}'CY} dY \\ &= -CK_1(\mathcal{B}'C) + K_0(\mathcal{B}'C). \end{aligned} \tag{B17}$$

Thus, for any real constant $\mathcal{B}$, positive or negative, we find

$$R(\omega) = 2A\mathrm{e}^{\mathcal{B}}\left\{C\frac{|\mathcal{B}|}{\mathcal{B}}K_1(|\mathcal{B}|C) + K_0(|\mathcal{B}|C)\right\} = 2\left(\frac{1}{\omega} + R_2(\omega)\right), \tag{B18}$$

$$A = \alpha/2, \quad C = \sqrt{1+\nu}, \quad \mathcal{B} = A\omega = \frac{\alpha}{2}(s-x). \tag{B19}$$

APPENDIX C

The Antiplane Shear Problem for a Graded Medium Containing a Crack

It can be shown that the integral equation corresponding to an infinite inhomogeneous isotropic medium containing a crack and subjected to anti-plane shear loading is given by

$$\frac{1}{\pi}\int_{-a}^{a}\left[\frac{1}{s-x}+M_A(x,s)\right]g(s)\,ds=\frac{1}{\mu_0}\mathrm{e}^{-\alpha x}\sigma_{yz}(x,0), \qquad -a<x<a, \tag{C1}$$

$$g(x)=\frac{\partial w(x,0)}{\partial x}, \tag{C2}$$

$$M_A(x,s)=\frac{\alpha}{2}\mathrm{e}^{\mathcal{B}}\left\{\frac{|\mathcal{B}|}{\mathcal{B}}K_1(|\mathcal{B}|)+K_0(|\mathcal{B}|)\right\}-\frac{1}{s-x}, \tag{C3}$$

$$\mathcal{B}=\frac{\alpha}{2}(s-x) \tag{C4}$$

where $K_0(z)$ and $K_1(z)$ are the modified Bessel functions of order zero and one respectively. It is again assumed that the shear modulus of the medium is approximated by $\mu(x)=\mu_0\exp(\alpha x)$. Note that the kernel of the integral equation (C3) is nearly identical to that given by (20) for the in-plane problem. Table C1 shows some sample results obtained from (C1) and their comparison with that obtained by delale (1985) numerically. Since the kernel is in closed form, (C1) can be solved within any desired degree of accuracy.

Table C1 The normalized mode III stress intensity factor for an inhomogeneous isotropic medium subjected to uniform crack surface traction, $\sigma_{yz}(x,0)=-p_0, -a<x<a$.

	Delale [1985]		Present	
αa	$\frac{k_3(a)}{p_0\sqrt{a}}$	$\frac{k_3(-a)}{p_0\sqrt{a}}$	$\frac{k_3(a)}{p_0\sqrt{a}}$	$\frac{k_3(-a)}{p_0\sqrt{a}}$
0.0	1.0	1.0	1.0	1.0
0.1	1.024	0.973	1.0228	0.9731
0.2	1.045	0.944	1.0427	0.9443
0.3	1.063	0.914	1.0605	0.9149
0.4	1.080	0.884	1.0763	0.8857
0.5	1.095	0.855	1.0906	0.8570
0.75	1.127	0.787	1.1205	0.7894
1.00	1.153	0.726	1.1438	0.7291

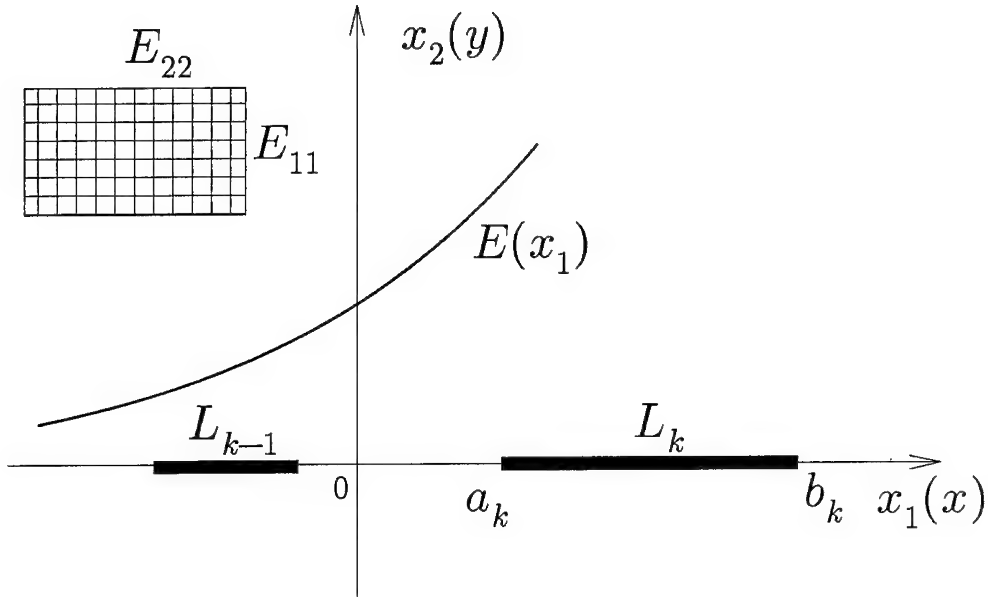

Figure 1 Geometry and notation of the collinear crack problem

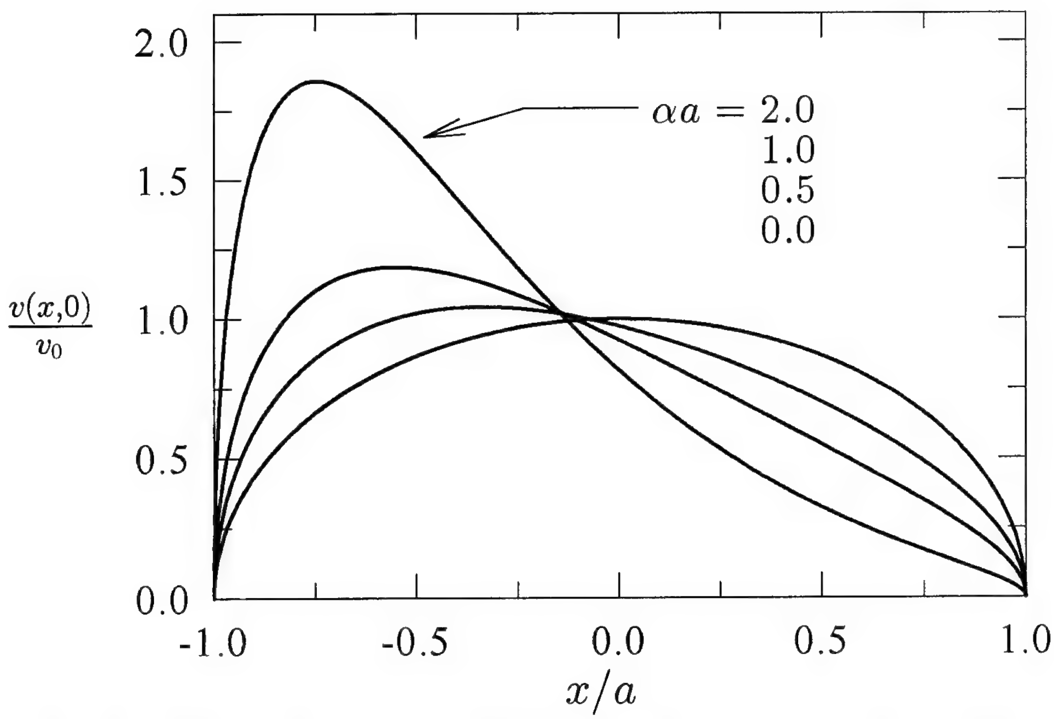

Figure 2 Crack surface displacements $v(x, 0)$ in an inhomogeneous isotropic medium under uniform pressure p_0 applied to the crack surfaces. ($v_0 = 2ap_0(1-\nu^2)/E_0$, $\nu = 0.3$, plane strain conditions.)

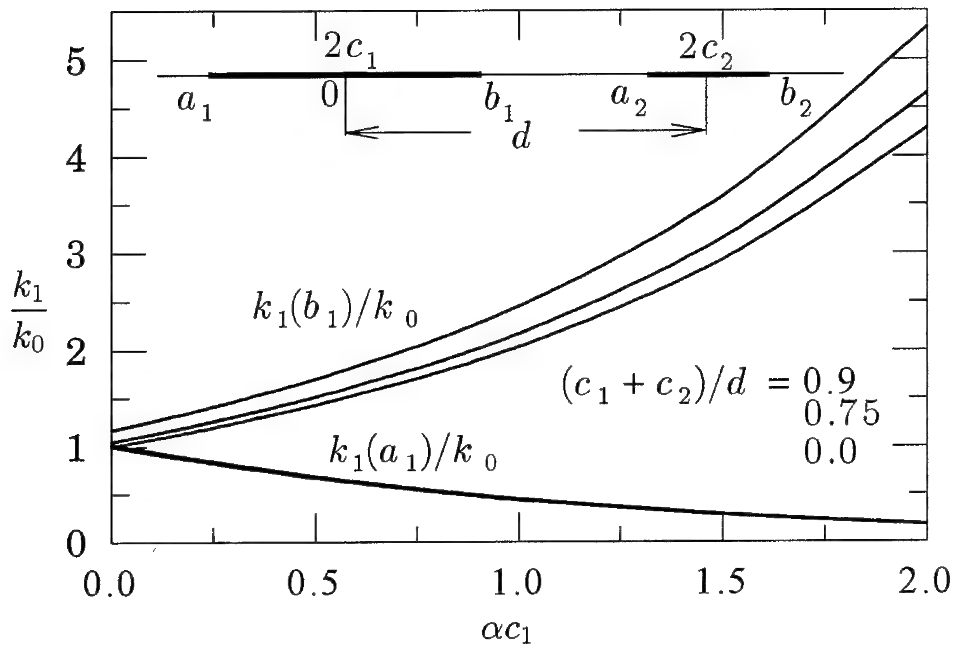

Figure 3 Stress intensity factors at the tips a_1 and b_1 of two unequal cracks in an isotropic FGM under fixed-grip condition, $\sigma_{yy}(x,0) = -\widehat{E}_0\varepsilon_0\exp(\alpha x)$, $k_0 = \widehat{E}_0\varepsilon_0\sqrt{c_1}$, $\widehat{E}_0 = E_0/(1-\nu^2)$, $c_2/c_1 = 0.25$, $\nu = 0.3$.

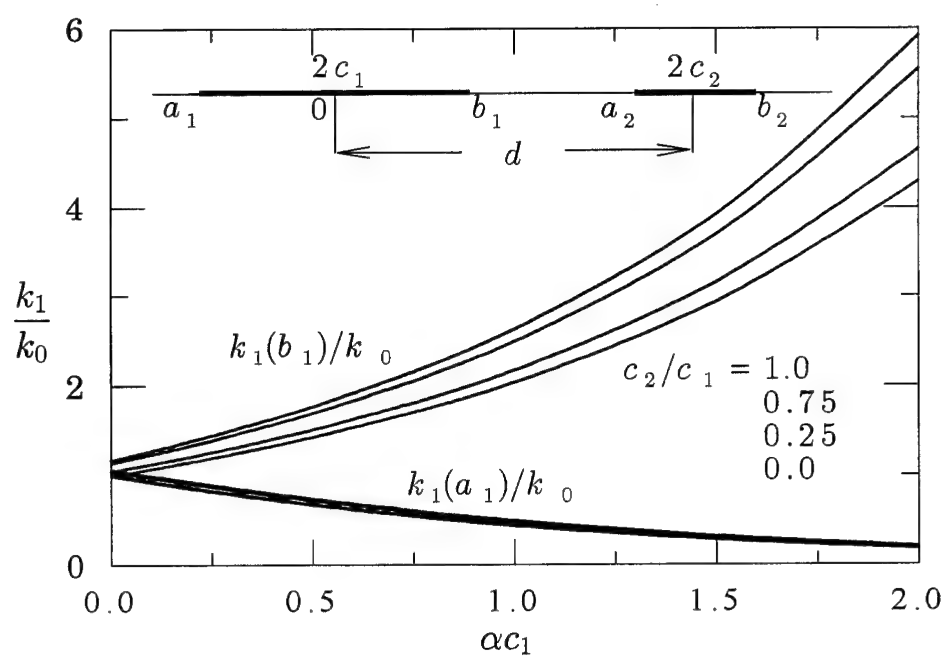

Figure 4 Stress intensity factors at the tips a_1 and b_1 of two unequal cracks in an isotropic FGM under fixed-grip condition, $\sigma_{yy}(x,0) = -\widehat{E}_0\varepsilon_0\exp(\alpha x)$, $k_0 = \widehat{E}_0\varepsilon_0\sqrt{c_1}$, $\widehat{E}_0 = E_0/(1-\nu^2)$, $(c_2 + c_1)/d = 0.75$, $\nu = 0.3$.

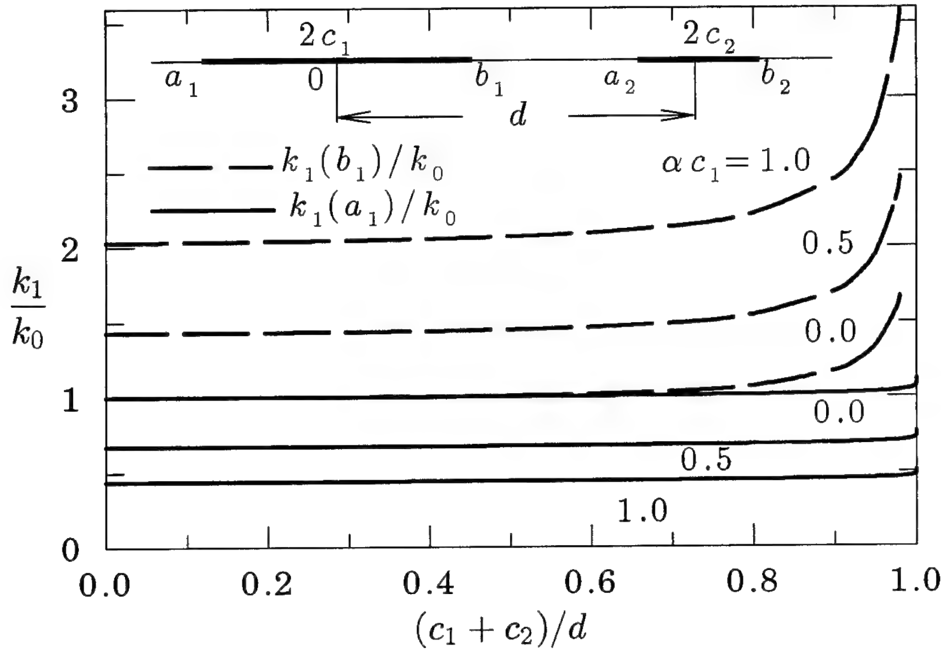

Figure 5 Stress intensity factors at the tips a_1 and b_1 of two unequal cracks in an isotropic FGM under fixed-grip condition, $\sigma_{yy}(x,0) = -\widehat{E}_0\varepsilon_0\exp(\alpha x)$, $k_0 = \widehat{E}_0\varepsilon_0\sqrt{c_1}$, $\widehat{E}_0 = E_0/(1-\nu^2)$, $c_2/c_1 = 0.25$, $\nu = 0.3$.

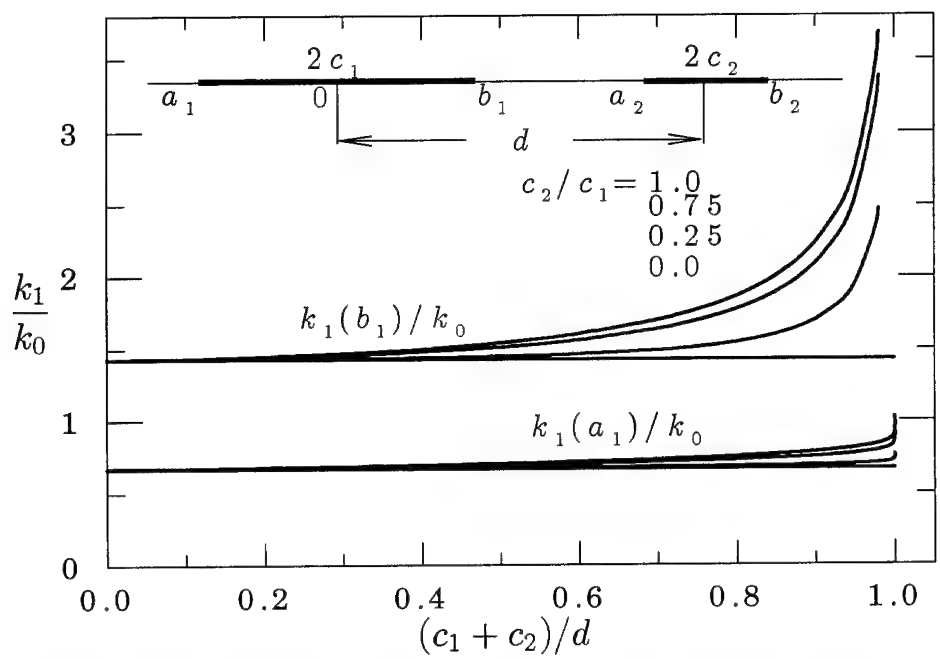

Figure 6 Stress intensity factors at the tips a_1 and b_1 of two unequal cracks in an isotropic FGM under fixed-grip condition, $\sigma_{yy}(x,0) = -\widehat{E}_0\varepsilon_0\exp(\alpha x)$, $k_0 = \widehat{E}_0\varepsilon_0\sqrt{c_1}$, $\widehat{E}_0 = E_0/(1-\nu^2)$, $\alpha c_1 = 0.5$, $\nu = 0.3$.

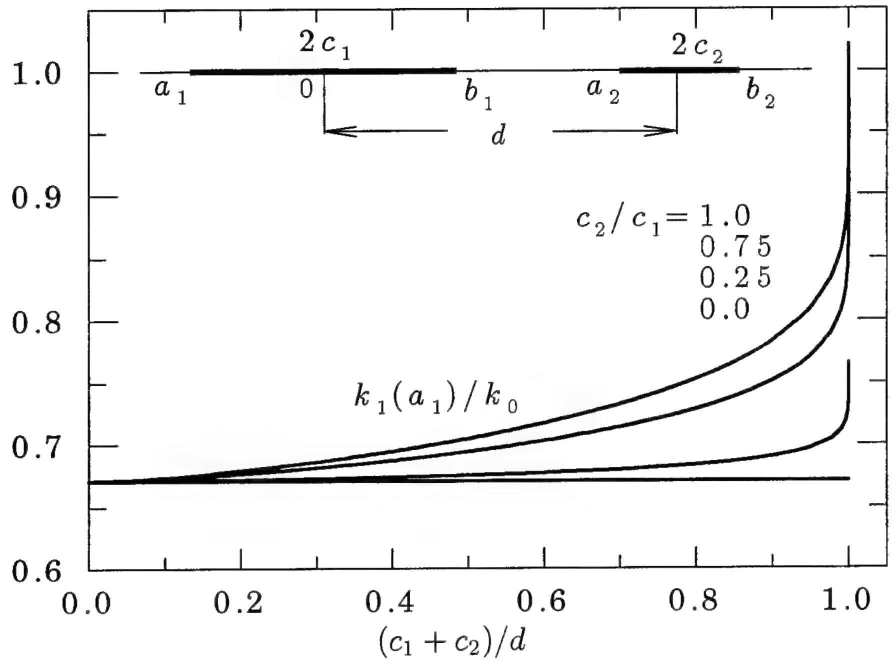

Figure 7 Stress intensity factors at the tip a_1 of two unequal cracks in an isotropic FGM under fixed-grip condition, $\sigma_{yy}(x,0) = -\widehat{E}_0\varepsilon_0 \exp(\alpha x)$, $k_0 = \widehat{E}_0\varepsilon_0\sqrt{c_1}$, $\widehat{E}_0 = E_0/(1-\nu^2)$, $\alpha c_1 = 0.5$, $\nu = 0.3$.

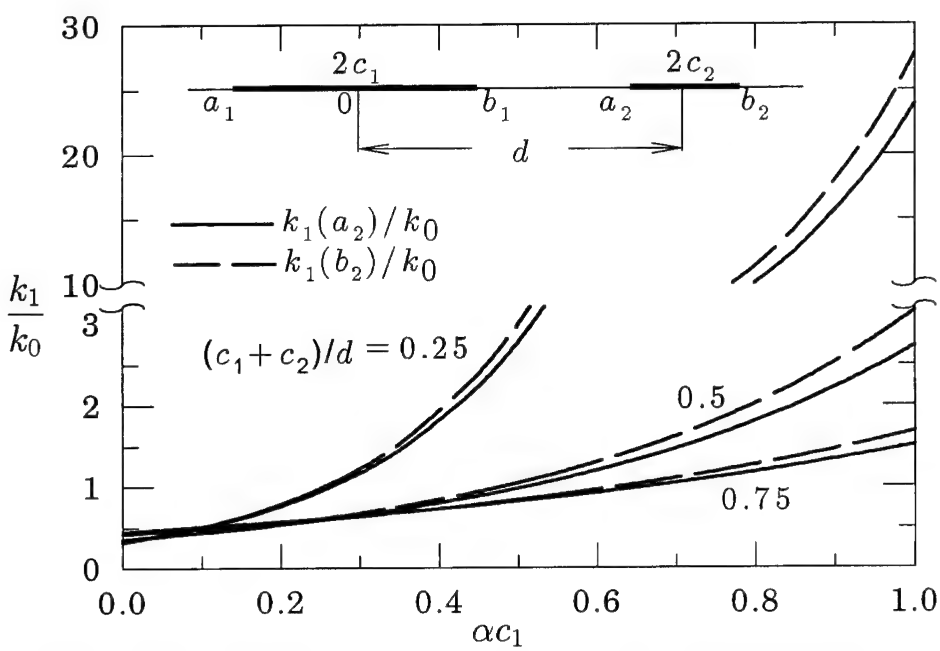

Figure 8 Stress intensity factors at the tips a_2 and b_2 of two unequal cracks in an isotropic FGM under fixed-grip condition, $\sigma_{yy}(x,0) = -\widehat{E}_0\varepsilon_0 \exp(\alpha x)$, $k_0 = \widehat{E}_0\varepsilon_0\sqrt{c_1}$, $\widehat{E}_0 = E_0/(1-\nu^2)$, $c_2/c_1 = 0.1$, $\nu = 0.3$.

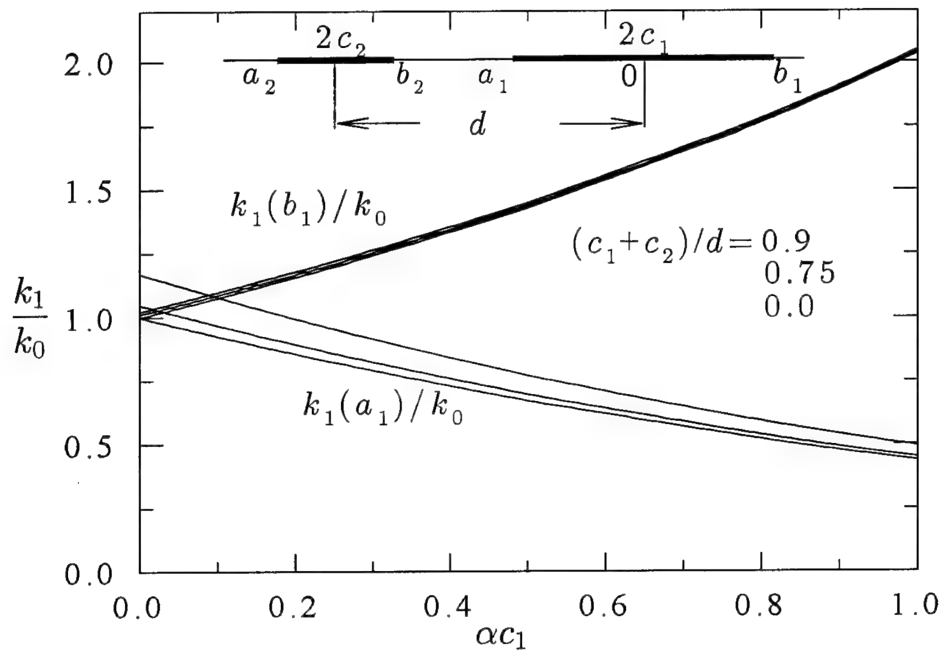

Figure 9 Stress intensity factors at the tips a_1 and b_1 of two unequal cracks in an isotropic FGM under fixed-grip condition, $\sigma_{yy}(x,0) = -\widehat{E}_0\varepsilon_0\exp(\alpha x)$, $k_0 = \widehat{E}_0\varepsilon_0\sqrt{c_1}$, $\widehat{E}_0 = E_0/(1-\nu^2)$, $c_2/c_1 = 0.25$, $\nu = 0.3$.

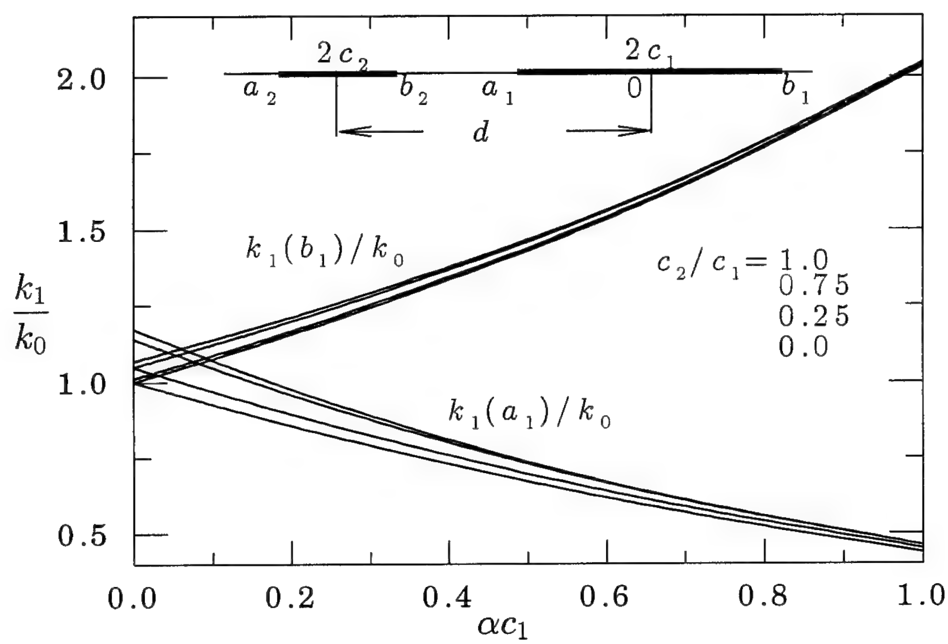

Figure 10 Stress intensity factors at the tips a_1 and b_1 of two unequal cracks in an isotropic FGM under fixed-grip condition, $\sigma_{yy}(x,0) = -\widehat{E}_0\varepsilon_0\exp(\alpha x)$, $k_0 = \widehat{E}_0\varepsilon_0\sqrt{c_1}$, $\widehat{E}_0 = E_0/(1-\nu^2)$, $(c_2+c_1)/d_2 = 0.75$, $\nu = 0.3$.

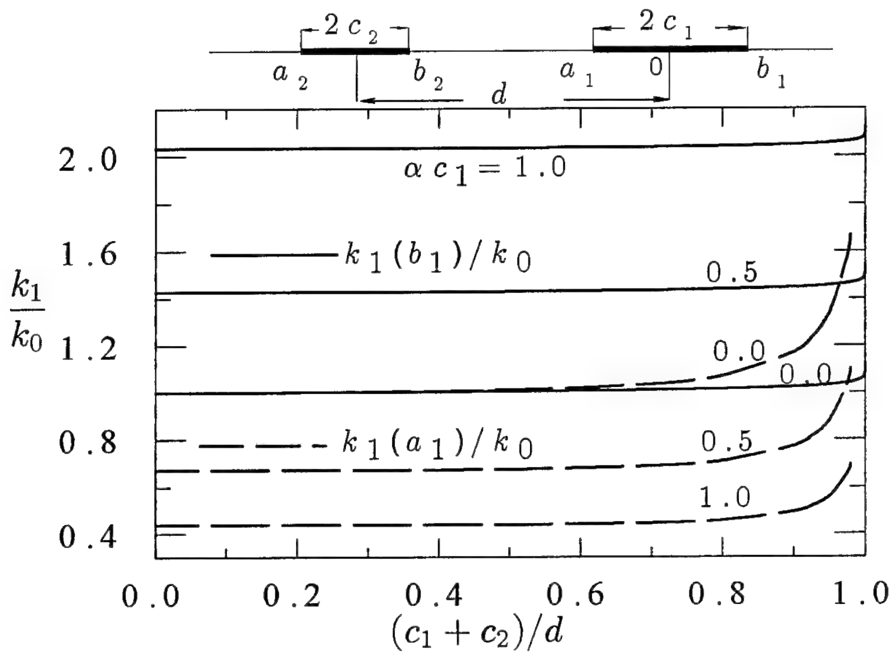

Figure 11 Stress intensity factors at the tips a_1 and b_1 of two unequal cracks in an isotropic FGM under fixed-grip condition, $\sigma_{yy}(x,0) = -\widehat{E}_0\varepsilon_0\exp(\alpha x)$, $k_0 = \widehat{E}_0\varepsilon_0\sqrt{c_1}$, $\widehat{E}_0 = E_0/(1-\nu^2)$, $c_2/c_1 = 0.25$, $\nu = 0.3$.

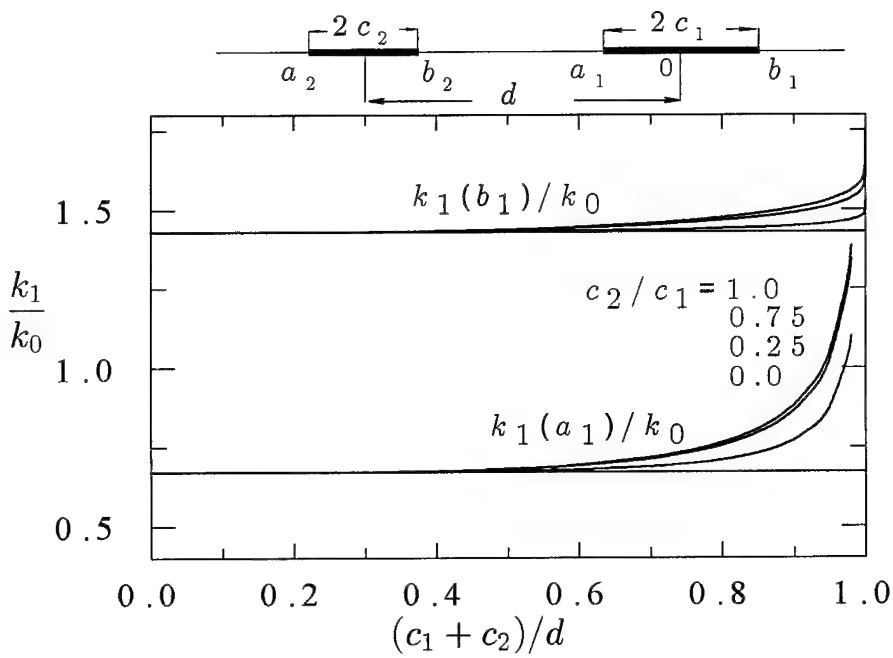

Figure 12 Stress intensity factors at the tips a_1 and b_1 of two unequal cracks in an isotropic FGM under fixed-grip condition, $\sigma_{yy}(x,0) = -\widehat{E}_0\varepsilon_0\exp(\alpha x)$, $k_0 = \widehat{E}_0\varepsilon_0\sqrt{c_1}$, $\widehat{E}_0 = E_0/(1-\nu^2)$, $\alpha c_1 = 0.5$, $\nu = 0.3$.

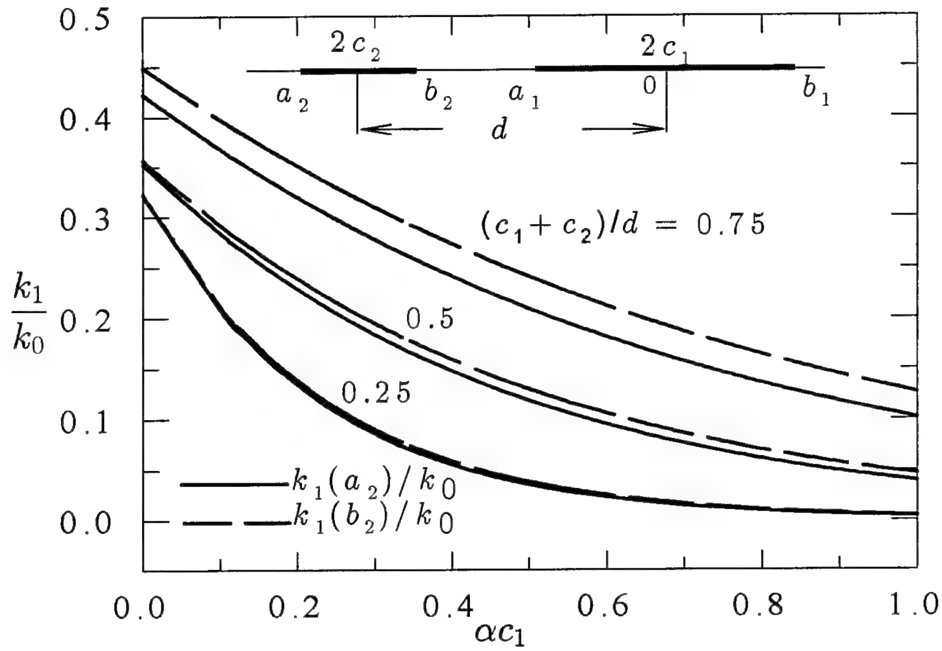

Figure 13 Stress intensity factors at the tips a_2 and b_2 of two unequal cracks in an isotropic FGM under fixed-grip condition, $\sigma_{yy}(x,0) = -\widehat{E}_0\varepsilon_0\exp(\alpha x)$, $k_0 = \widehat{E}_0\varepsilon_0\sqrt{c_1}$, $\widehat{E}_0 = E_0/(1-\nu^2)$, $c_2/c_1 = 0.1$, $\nu = 0.3$.

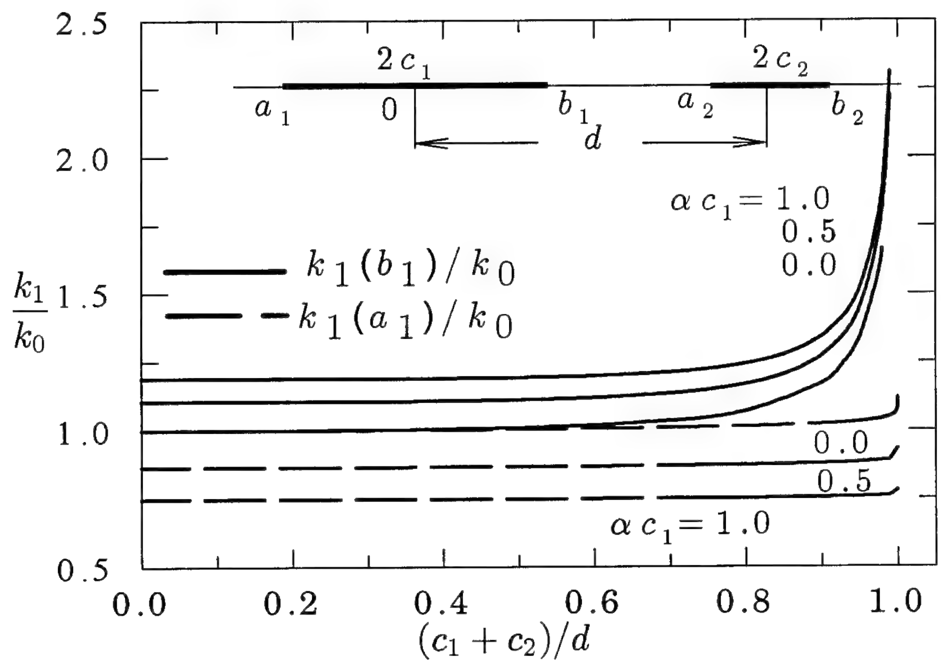

Figure 14 Stress intensity factors at the tips a_1 and b_1 of two unequal cracks in an isotropic FGM under fixed-load condition, $\sigma_{yy}(x,0) = -p_0$, $k_0 = p_0\sqrt{c_1}$, $c_2/c_1 = 0.25$, $\nu = 0.3$.

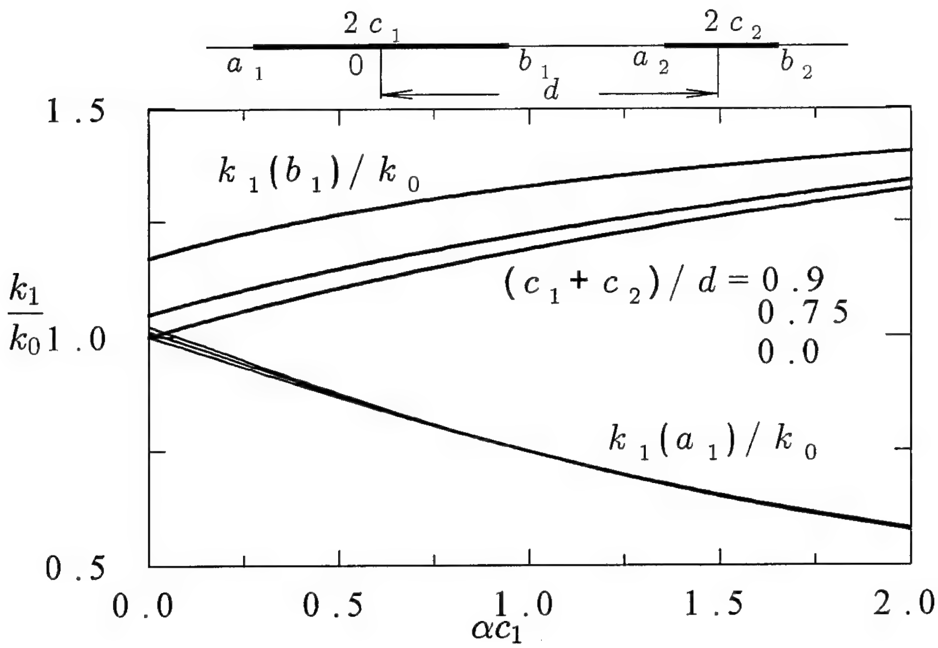

Figure 15 Stress intensity factors at the tips a_1 and b_1 of two unequal cracks in an isotropic FGM under fixed-load condition, $\sigma_{yy}(x,0) = -p_0$, $k_0 = p_0\sqrt{c_1}$, $c_2/c_1 = 0.25$, $\nu = 0.3$.

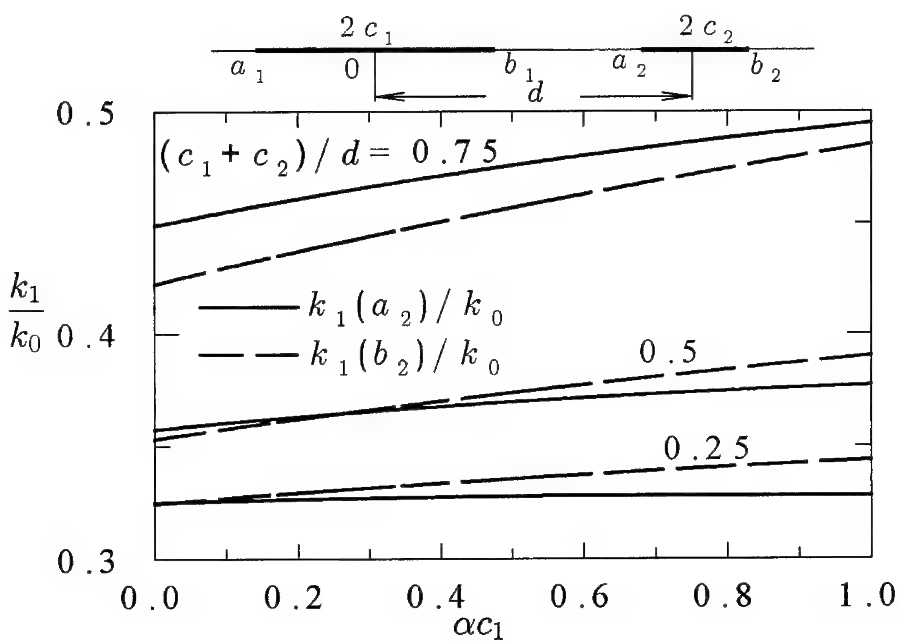

Figure 16 Stress intensity factors at the tips a_2 and b_2 of two unequal cracks in an isotropic FGM under fixed-load condition, $\sigma_{yy}(x,0) = -p_0$, $k_0 = p_0\sqrt{c_1}$, $c_2/c_1 = 0.1$, $\nu = 0.3$.

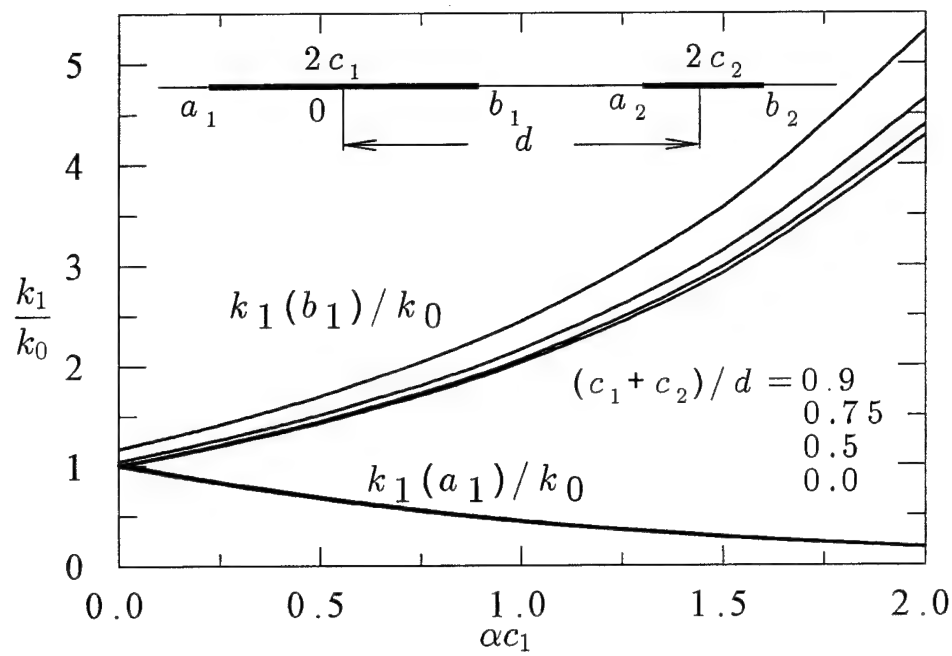

Figure 17 Stress intensity factors at the tips a_1 and b_1 of two unequal cracks in an orthotropic FGM under fixed-grip condition, $\sigma_{yy}(x,0) = -\varepsilon_0 E_0/\delta_0^2 \exp(\alpha x_1)$, $k_0 = (\varepsilon_0 E_0/\delta_0^2)\sqrt{c_1}$, $c_2/c_1 = 0.25$, $\kappa = 0.5$, $\nu = 0.3$.

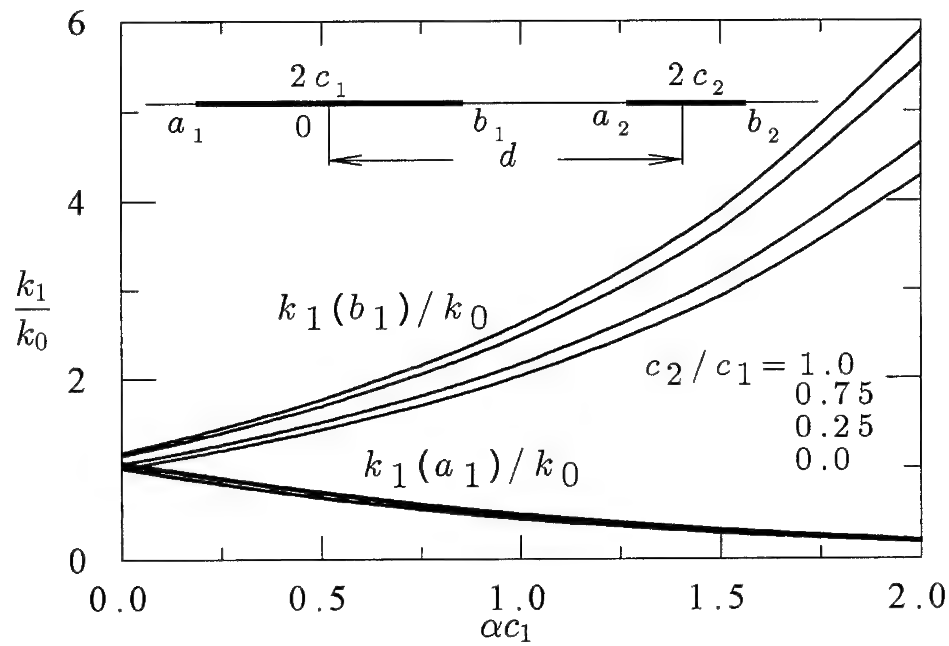

Figure 18 Stress intensity factors at the tips a_1 and b_1 of two unequal cracks in an orthotropic FGM under fixed-grip condition, $\sigma_{yy}(x,0) = -\varepsilon_0 E_0/\delta_0^2 \exp(\alpha x_1)$, $k_0 = (\varepsilon_0 E_0/\delta_0^2)\sqrt{c_1}$, $(c_2 + c_1)/d = 0.75$, $\kappa = 0.5$, $\nu = 0.3$.

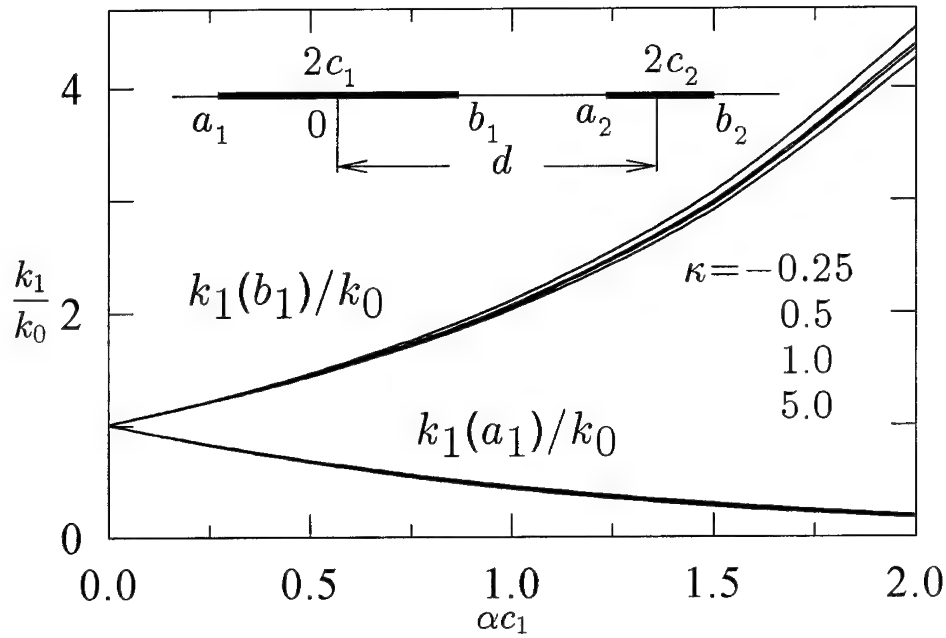

Figure19 Stress intensity factors at the tips a_1 and $b_1$of two unequal cracks in an orthotropic FGM under fixed-grip condition, $\sigma_{yy}(x,0) = -E_0\varepsilon_0/\delta_0^2\exp(\alpha x_1)$, $k_0 = (E_0\varepsilon_0/\delta_0^2)\sqrt{c_1}$, $(c_2 + c_1)/d = 0.5$, $c_2/c_1 = 0.25$, $\nu = 0.3$.

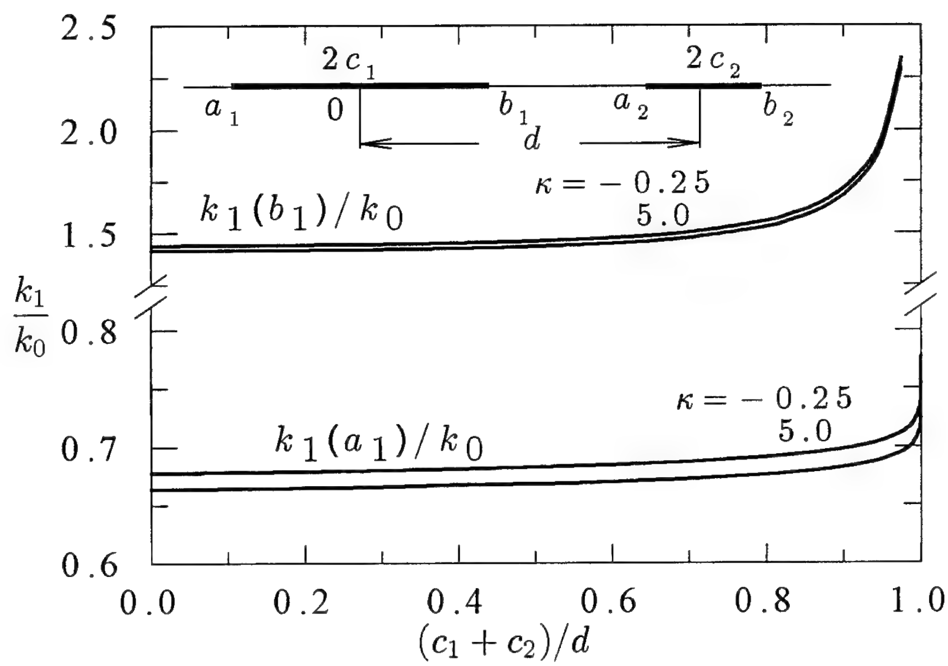

Figure 20 Stress intensity factors at the tips a_1 and b_1 of two unequal cracks in an orthotropic FGM under fixed-grip condition, $\sigma_{yy}(x,0) = -E_0\varepsilon_0/\delta_0^2\exp(\alpha x_1)$, $k_0 = (E_0\varepsilon_0/\delta_0^2)\sqrt{c_1}$, $\alpha c_1 = 0.5$, $c_2/c_1 = 0.25$, $\nu = 0.3$.

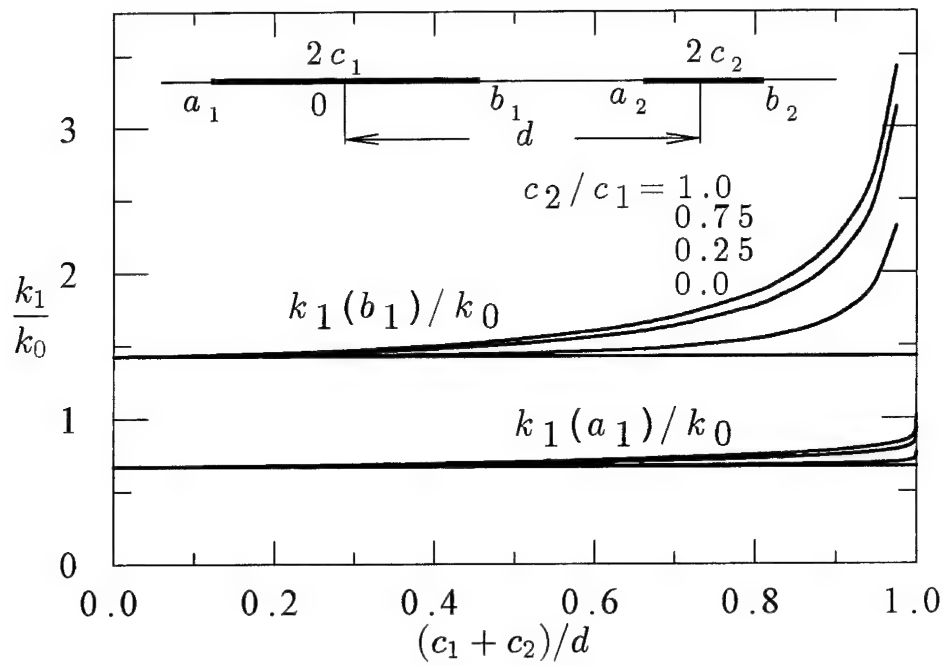

Figure 21 Stress intensity factors at the tips a_1 and b_1 of two unequal cracks in an orthotropic FGM under fixed-grip condition, $\sigma_{yy}(x,0) = -E_0\varepsilon_0/\delta^2\exp(\alpha x_1)$, $k_0 = E_0\varepsilon_0/\delta^2\sqrt{c_1}$, $\alpha c_1 = 0.5$, $\nu = 0.3$, $\kappa = 0.5$.

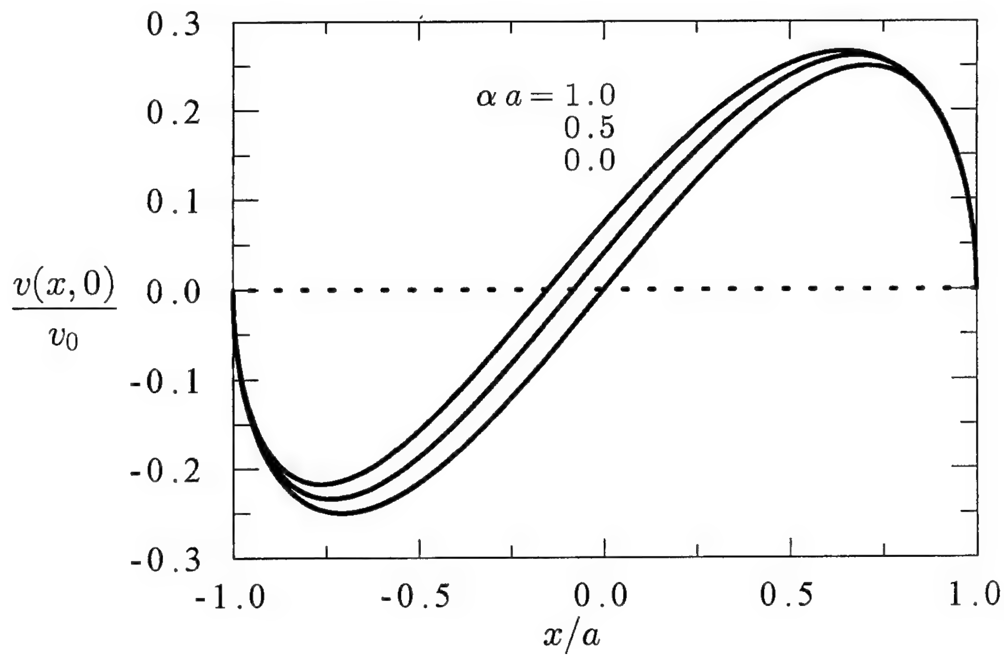

Figure 22 Crack surface displacement for an isotropic inhomogeneous medium under fixed-grip condition, disregarding interpenetration of crack surfaces, $\sigma_{yy}(x,0) = -\widehat{E}_0\varepsilon_1(x_1/a)\exp(\alpha x_1)$, $v_0 = 2\varepsilon_1 a$, $\widehat{E}_0 = E_0/(1-\nu^2)$, $\nu = 0.3$.

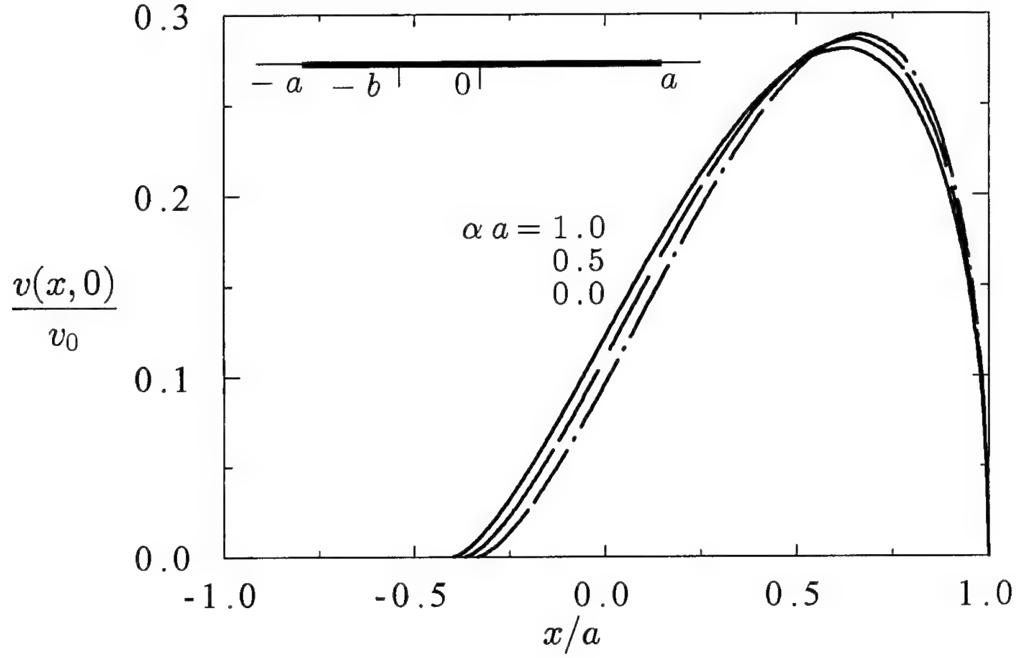

Figure 23 Crack surface displacement for an isotropic inhomogeneous medium under fixed-grip condition in the presence of smooth contact, $\sigma_{yy}(x,0) = -\widehat{E}_0\varepsilon_1(x/a)\exp(\alpha x)$, $v_0 = 2\varepsilon_1 a$, $\widehat{E}_0 = E_0/(1-\nu^2), \nu = 0.3$, $\varepsilon_1 > 0$.

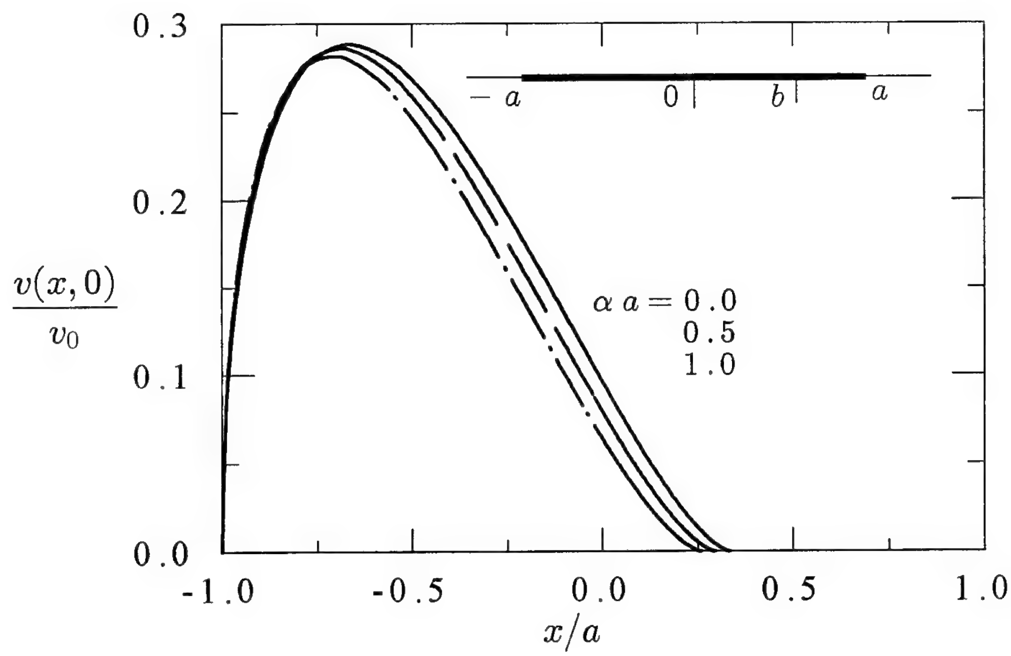

Figure 24 Crack surface displacement for an isotropic inhomogeneous medium under fixed-grip condition in the presence of smooth contact, $\sigma_{yy}(x,0) = -\widehat{E}_0\varepsilon_1(x/a)\exp(\alpha x)$, $v_0 = 2\varepsilon_1 a$, $\widehat{E}_0 = E_0/(1-\nu^2), \nu = 0.3, \varepsilon_1 < 0$.

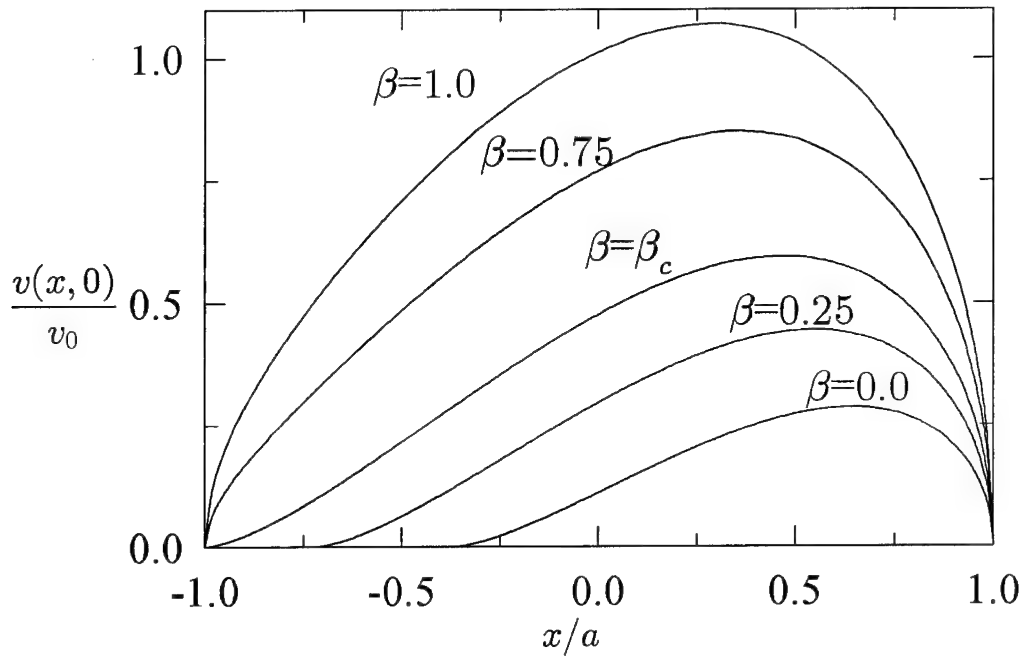

Figure25 Crack surface displacements for an infinite isotropic inhomogeneous medium under fixed-grip condition : $\varepsilon_{yy} = \varepsilon_0 + \varepsilon_1(x/a)$, $\varepsilon_0/\varepsilon_1 = \beta$, $\beta_c = 0.4485$, $\alpha a = 0.5$, $\nu = 0.3$.

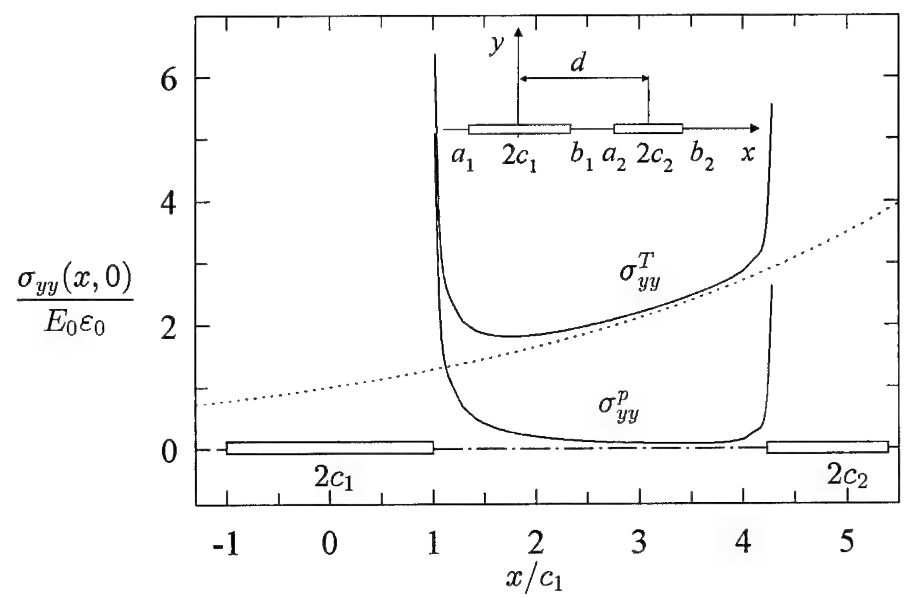

Figure 26 The stress distribution in the plane of the crack in an inhomogeneous isotropic medium with two unequal colinear cracks under the remote loading $\sigma_{yy}^{\infty}(x,0) = \varepsilon_0 E_0 \mathrm{e}^{\alpha x}$, $\alpha c_1 = 0.25$, $c_2/c_1 = 0.1$, $(c_1 + c_2)/d = 0.25$, $\sigma_{yy}^T(x,0)$: the total stress, $\sigma_{yy}^p(x,0)$: the perturbation, $\sigma_{yy}^{\infty}(x,0)$: stress at infinity (the applied stress), $\sigma_{yy}^T = \sigma_{yy}^p + \sigma_{yy}^{\infty}$.

Table 1 The normalized stress intensity factors for an inhomogeneous isotropic medium under crack surface tractions given by Eq. 27 ($\nu = 0.3$).

Plane Strain								
αa	$\frac{k_1(a)}{p_0\sqrt{a}}$	$\frac{k_1(-a)}{p_0\sqrt{a}}$	$\frac{k_1(a)}{p_1\sqrt{a}}$	$\frac{k_1(-a)}{p_1\sqrt{a}}$	$\frac{k_1(a)}{p_2\sqrt{a}}$	$\frac{k_1(-a)}{p_2\sqrt{a}}$	$\frac{k_1(a)}{p_3\sqrt{a}}$	$\frac{k_1(-a)}{p_3\sqrt{a}}$
0.00	1.0	1.0	0.5	−0.5	0.5	0.5	0.375	−0.375
0.01	1.0025	0.9975	0.5000	−0.5000	0.5006	0.4994	0.3750	−0.3750
0.10	1.0238	0.9740	0.4998	−0.4998	0.5060	0.4935	0.3749	−0.3749
0.25	1.0567	0.9334	0.4989	−0.4986	0.5142	0.4833	0.3745	−0.3743
0.50	1.1062	0.8667	0.4962	−0.4944	0.5267	0.4665	0.3731	−0.3722
0.75	1.1504	0.8044	0.4923	−0.4878	0.5379	0.4505	0.3711	−0.3689
1.00	1.1902	0.7480	0.4876	−0.4793	0.5483	0.4357	0.3688	−0.3646
1.50	1.2598	0.6536	0.4771	−0.4590	0.5668	0.4098	0.3634	−0.3542
2.00	1.3195	0.5810	0.4660	−0.4371	0.5833	0.3881	0.3580	−0.3429
Plane Stress								
αa	$\frac{k_1(a)}{p_0\sqrt{a}}$	$\frac{k_1(-a)}{p_0\sqrt{a}}$	$\frac{k_1(a)}{p_1\sqrt{a}}$	$\frac{k_1(-a)}{p_1\sqrt{a}}$	$\frac{k_1(a)}{p_2\sqrt{a}}$	$\frac{k_1(-a)}{p_2\sqrt{a}}$	$\frac{k_1(a)}{p_3\sqrt{a}}$	$\frac{k_1(-a)}{p_3\sqrt{a}}$
0.00	1.0	1.0	0.5	−0.5	0.5	0.5	0.375	−0.375
0.01	1.0025	0.9975	0.5000	−0.5000	0.5006	0.4994	0.3750	−0.3750
0.10	1.0235	0.9737	0.4998	−0.4998	0.5059	0.4934	0.3749	−0.3749
0.25	1.0553	0.9324	0.4989	−0.4985	0.5139	0.4831	0.3745	−0.3720
0.50	1.1019	0.8640	0.4962	−0.4941	0.5256	0.4658	0.3731	−0.3720
0.75	1.1421	0.8002	0.4923	−0.4870	0.5359	0.4494	0.3712	−0.3685
1.00	1.1774	0.7427	0.4879	−0.4781	0.5451	0.4343	0.3689	−0.3640
1.50	1.2369	0.6473	0.4780	−0.4570	0.5611	0.4079	0.3639	−0.3532
2.00	1.2862	0.5746	0.4680	−0.4345	0.5749	0.3859	0.3590	−0.3415

Table 2 The normalized stress intensity factors for an inhomogeneous isotropic medium under fixed - grip condition ($\widetilde{E}_0 = E_0$ for plane stress, $\widetilde{E}_0 = E_0/(1-\nu^2)$ for plane strain , $\nu = 0.3$, see Eq. 30).

$\nu = 0.3$	Plane Strain				Plane Stress			
αa	$\frac{k_1(a)}{\epsilon_0\widetilde{E}_0\sqrt{a}}$	$\frac{k_1(-a)}{\epsilon_0\widetilde{E}_0\sqrt{a}}$	$\frac{k_1(a)}{\epsilon_1\widetilde{E}_0\sqrt{a}}$	$\frac{k_1(-a)}{\epsilon_1\widetilde{E}_0\sqrt{a}}$	$\frac{k_1(a)}{\epsilon_0\widetilde{E}_0\sqrt{a}}$	$\frac{k_1(-a)}{\epsilon_0\widetilde{E}_0\sqrt{a}}$	$\frac{k_1(a)}{\epsilon_1\widetilde{E}_0\sqrt{a}}$	$\frac{k_1(-a)}{\epsilon_1\widetilde{E}_0\sqrt{a}}$
0.00	1.0	1.0	0.5	−0.5	1.0	1.0	0.5	−0.5
0.01	1.0075	0.9925	0.5050	−0.4950	1.0075	0.9925	0.5050	−0.4950
0.10	1.0764	0.9264	0.5523	−0.4523	1.0761	0.9261	0.5523	−0.4522
0.25	1.1986	0.8230	0.6402	−0.3886	1.1972	0.8219	0.6401	−0.3885
0.50	1.4290	0.6710	0.8151	−0.3010	1.4245	0.6683	0.8146	−0.3009
0.75	1.7029	0.5434	1.0326	−0.2325	1.6940	0.5395	1.0311	−0.2326
1.00	2.0332	0.4379	1.3029	−0.1794	2.0189	0.4331	1.2997	−0.1795
1.50	2.9294	0.2811	2.0570	−0.1064	2.9012	0.2760	2.0482	−0.1067
2.00	4.2933	0.1786	3.2297	−0.0631	4.2454	0.1741	3.2107	−0.0633

Table 3 The effect of Poison's ratio on the normalized stress intensity factors for an inhomogeneous isotropic medium ($\widetilde{E}_0 = E_0$ for plane stress, $\widetilde{E}_0 = E_0/(1-\nu^2)$ for plane strain, see Eqs. 28 and 30).

Plane Strain								
$\alpha a = 0.5$	Fixed-Grip				Fixed-Load			
ν	$\frac{k_1(a)}{\epsilon_0 \widetilde{E}_0 \sqrt{a}}$	$\frac{k_1(-a)}{\epsilon_0 \widetilde{E}_0 \sqrt{a}}$	$\frac{k_1(a)}{\epsilon_1 \widetilde{E}_0 \sqrt{a}}$	$\frac{k_1(-a)}{\epsilon_1 \widetilde{E}_0 \sqrt{a}}$	$\frac{k_1(a)}{p_0 \sqrt{a}}$	$\frac{k_1(-a)}{p_0 \sqrt{a}}$	$\frac{k_1(a)}{p_1 \sqrt{a}}$	$\frac{k_1(-a)}{p_1 \sqrt{a}}$
0.0	1.4129	0.6615	0.8131	−0.3009	1.0906	0.8570	0.4961	−0.4932
0.1	1.4174	0.6641	0.8137	−0.3009	1.0950	0.8597	0.4961	−0.4935
0.2	1.4227	0.6672	0.8143	−0.3009	1.1001	0.8629	0.4962	−0.4939
0.3	1.4290	0.6710	0.8151	−0.3010	1.1062	0.8667	0.4962	−0.4944
0.4	1.4368	0.6755	0.8162	−0.3010	1.1137	0.8715	0.4963	−0.4950
0.5	1.4468	0.6814	0.8176	−0.3011	1.1234	0.8775	0.4964	−0.4959
Plane Stress								
$\alpha a = 0.5$	Fixed-Grip				Fixed-Load			
ν	$\frac{k_1(a)}{\epsilon_0 \widetilde{E}_0 \sqrt{a}}$	$\frac{k_1(-a)}{\epsilon_0 \widetilde{E}_0 \sqrt{a}}$	$\frac{k_1(a)}{\epsilon_1 \widetilde{E}_0 \sqrt{a}}$	$\frac{k_1(-a)}{\epsilon_1 \widetilde{E}_0 \sqrt{a}}$	$\frac{k_1(a)}{p_0 \sqrt{a}}$	$\frac{k_1(-a)}{p_0 \sqrt{a}}$	$\frac{k_1(a)}{p_1 \sqrt{a}}$	$\frac{k_1(-a)}{p_1 \sqrt{a}}$
0.0	1.4129	0.6615	0.8131	−0.3009	1.0906	0.8570	0.4961	−0.4932
0.1	1.4170	0.6639	0.8136	−0.3009	1.0946	0.8594	0.4961	−0.4935
0.2	1.4208	0.6661	0.8141	−0.3009	1.0983	0.8618	0.4961	−0.4938
0.3	1.4245	0.6683	0.8146	−0.3009	1.1019	0.8640	0.4962	−0.4941
0.4	1.4280	0.6704	0.8150	−0.3009	1.1053	0.8661	0.4962	−0.4943
0.5	1.4314	0.6724	0.8155	−0.3010	1.1085	0.8682	0.4962	−0.4946

Plane Strain								
$\alpha a = 1.0$	Fixed-Grip				Fixed-Load			
ν	$\frac{k_1(a)}{\epsilon_0 \widetilde{E}_0 \sqrt{a}}$	$\frac{k_1(-a)}{\epsilon_0 \widetilde{E}_0 \sqrt{a}}$	$\frac{k_1(a)}{\epsilon_1 \widetilde{E}_0 \sqrt{a}}$	$\frac{k_1(-a)}{\epsilon_1 \widetilde{E}_0 \sqrt{a}}$	$\frac{k_1(a)}{p_0 \sqrt{a}}$	$\frac{k_1(-a)}{p_0 \sqrt{a}}$	$\frac{k_1(a)}{p_1 \sqrt{a}}$	$\frac{k_1(-a)}{p_1 \sqrt{a}}$
0.0	1.9819	0.4208	1.2917	−0.1798	1.1438	0.7291	0.4888	−0.4752
0.1	1.9963	0.4256	1.2948	−0.1797	1.1569	0.7344	0.4884	−0.4763
0.2	2.0132	0.4312	1.2984	−0.1795	1.1722	0.7406	0.4880	−0.4777
0.3	2.0332	0.4379	1.3029	−0.1794	1.1902	0.7480	0.4876	−0.4793
0.4	2.0576	0.4459	1.3084	−0.1793	1.2121	0.7569	0.4874	−0.4813
0.5	2.0885	0.4560	1.3157	−0.1793	1.2396	0.7683	0.4873	−0.4840
Plane Stress								
$\alpha a = 1.0$	Fixed-Grip				Fixed-Load			
ν	$\frac{k_1(a)}{\epsilon_0 \widetilde{E}_0 \sqrt{a}}$	$\frac{k_1(-a)}{\epsilon_0 \widetilde{E}_0 \sqrt{a}}$	$\frac{k_1(a)}{\epsilon_1 \widetilde{E}_0 \sqrt{a}}$	$\frac{k_1(-a)}{\epsilon_1 \widetilde{E}_0 \sqrt{a}}$	$\frac{k_1(a)}{p_0 \sqrt{a}}$	$\frac{k_1(-a)}{p_0 \sqrt{a}}$	$\frac{k_1(a)}{p_1 \sqrt{a}}$	$\frac{k_1(-a)}{p_1 \sqrt{a}}$
0.0	1.9819	0.4208	1.2917	−0.1798	1.1438	0.7291	0.4888	−0.4752
0.1	1.9949	0.4251	1.2945	−0.1797	1.1556	0.7339	0.4884	−0.4762
0.2	2.0072	0.4292	1.2971	−0.1796	1.1668	0.7384	0.4881	−0.4772
0.3	2.0189	0.4331	1.2997	−0.1795	1.1774	0.7427	0.4879	−0.4781
0.4	2.0301	0.4368	1.3022	−0.1794	1.1874	0.7468	0.4877	−0.4790
0.5	2.0407	0.4404	1.3046	−0.1793	1.1970	0.7507	0.4875	−0.4799

Table 4 The normalized stress intensity factors for an inhomogeneous isotropic medium under fixed - load condition for the case of plane strain ($\nu = 0.3$).

	Present				Delale and Erdogan [1983]			
αa	$\frac{k_1(a)}{p_0\sqrt{a}}$	$\frac{k_1(-a)}{p_0\sqrt{a}}$	$\frac{k_1(a)}{p_1\sqrt{a}}$	$\frac{k_1(-a)}{p_1\sqrt{a}}$	$\frac{k_1(a)}{p_0\sqrt{a}}$	$\frac{k_1(-a)}{p_0\sqrt{a}}$	$\frac{k_1(a)}{p_1\sqrt{a}}$	$\frac{k_1(-a)}{p_1\sqrt{a}}$
0.00	1.0	1.0	0.5	−0.5	1.0	1.0	0.5	−0.5
0.01	1.0025	0.9975	0.5000	−0.5000	1.003	0.997	0.500	−0.500
0.10	1.0238	0.9740	0.4998	−0.4998	1.026	0.973	0.500	−0.500
0.25	1.0567	0.9334	0.4989	−0.4986	1.061	0.931	0.498	−0.499
0.50	1.1062	0.8667	0.4962	−0.4944	1.117	0.863	0.494	−0.495
0.75	1.1504	0.8044	0.4923	−0.4878	1.170	0.801	0.489	−0.489
1.00	1.1902	0.7480	0.4876	−0.4793	1.222	0.745	0.483	−0.481

Table 5 The normalized stress intensity factors for an inhomogeneous isotropic medium under fixed - load condition for the case of plane stress ($\nu = 0.3$).

	Present				Delale and Erdogan [1983]			
αa	$\frac{k_1(a)}{p_0\sqrt{a}}$	$\frac{k_1(-a)}{p_0\sqrt{a}}$	$\frac{k_1(a)}{p_1\sqrt{a}}$	$\frac{k_1(-a)}{p_1\sqrt{a}}$	$\frac{k_1(a)}{p_0\sqrt{a}}$	$\frac{k_1(-a)}{p_0\sqrt{a}}$	$\frac{k_1(a)}{p_1\sqrt{a}}$	$\frac{k_1(-a)}{p_1\sqrt{a}}$
0.00	1.0	1.0	0.5	−0.5	1.0	1.0	0.5	−0.5
0.01	1.0025	0.9975	0.5000	−0.5000	1.003	0.997	0.500	−0.500
0.10	1.0235	0.9737	0.4998	−0.4998	1.025	0.973	0.500	−0.500
0.25	1.0553	0.9324	0.4989	−0.4985	1.060	0.930	0.498	−0.499
0.50	1.1019	0.8640	0.4962	−0.4941	1.113	0.861	0.495	−0.495
0.75	1.1421	0.8002	0.4923	−0.4870	1.162	0.797	0.489	−0.489
1.00	1.1774	0.7427	0.4879	−0.4781	1.209	0.740	0.483	−0.480

Table 6 The normalized stress intensity factors for an inhomogeneous isotropic medium under fixed - grip condition ($\widetilde{E}_0 = E_0$ for plane stress, $\widetilde{E}_0 = E_0/(1-\nu^2)$ for plane strain).

	Plane Strain				Plane Stress			
$\nu = 0.3$	Present		Delale and Erdogan [1983]		Present		Delale and Erdogan [1983]	
αa	$\frac{k_1(a)}{\epsilon_0\widetilde{E}_0\sqrt{a}}$	$\frac{k_1(-a)}{\epsilon_0\widetilde{E}_0\sqrt{a}}$	$\frac{k_1(a)}{\epsilon_0\widetilde{E}_0\sqrt{a}}$	$\frac{k_1(-a)}{\epsilon_0\widetilde{E}_0\sqrt{a}}$	$\frac{k_1(a)}{\epsilon_0\widetilde{E}_0\sqrt{a}}$	$\frac{k_1(-a)}{\epsilon_0\widetilde{E}_0\sqrt{a}}$	$\frac{k_1(a)}{\epsilon_0\widetilde{E}_0\sqrt{a}}$	$\frac{k_1(-a)}{\epsilon_0\widetilde{E}_0\sqrt{a}}$
0.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0.01	1.0075	0.9925	1.008	0.992	1.0075	0.9925	1.008	0.992
0.10	1.0764	0.9264	1.078	0.925	1.0761	0.9261	1.078	0.925
0.25	1.1986	0.8230	1.203	0.821	1.1972	0.8219	1.202	0.820
0.50	1.4290	0.6710	1.439	0.667	1.4245	0.6683	1.435	0.665
0.75	1.7029	0.5434	1.721	0.539	1.6940	0.5395	1.713	0.535
1.00	2.0332	0.4379	2.063	0.433	2.0189	0.4331	2.048	0.429

Table 7 Stress intensity factors at the tips $x = a_1$ and $x = b_1$ of two unequal cracks in an isotropic FGM for the case of plane strain, $\nu = 0.3$, $\widehat{E}_0 = E_0/(1-\nu^2)$.

Fixed-Grip	$\sigma_{yy}(x,0) = -\widehat{E}_0\varepsilon_0 \exp(\alpha x)$, $k_0 = \widehat{E}_0\varepsilon_0\sqrt{c_1}$					
$\alpha c_1 = 0.5$	$c_2/c_1 = 0.25$		$c_2/c_1 = 0.75$		$c_2/c_1 = 1.0$	
$(c_1+c_2)/d$	$\frac{k_1(a_1)}{k_0}$	$\frac{k_1(b_1)}{k_0}$	$\frac{k_1(a_1)}{k_0}$	$\frac{k_1(b_1)}{k_0}$	$\frac{k_1(a_1)}{k_0}$	$\frac{k_1(b_1)}{k_0}$
0.0	0.6710	1.4290	0.6710	1.4290	0.6710	1.4290
0.1	0.6713	1.4299	0.6728	1.4334	0.6735	1.4350
0.2	0.6720	1.4319	0.6766	1.4435	0.6791	1.4495
0.4	0.6740	1.4405	0.6873	1.4805	0.6947	1.5000
0.6	0.6773	1.4634	0.7030	1.5601	0.7171	1.6032
0.8	0.6835	1.5440	0.7284	1.7721	0.7526	1.8624
0.975	0.7034	2.3181	0.7883	3.1421	0.8314	3.4198
1.0	0.7656	∞	0.9395	∞	1.0210	∞

Fixed-Load	$\sigma_{yy}(x,0) = -p_0$, $k_0 = p_0\sqrt{c_1}$					
$\alpha c_1 = 0.5$	$c_2/c_1 = 0.25$		$c_2/c_1 = 0.75$		$c_2/c_1 = 1.0$	
$(c_1+c_2)/d$	$\frac{k_1(a_1)}{k_0}$	$\frac{k_1(b_1)}{k_0}$	$\frac{k_1(a_1)}{k_0}$	$\frac{k_1(b_1)}{k_0}$	$\frac{k_1(a_1)}{k_0}$	$\frac{k_1(b_1)}{k_0}$
0.0	0.8667	1.1062	0.8667	1.1062	0.8667	1.1062
0.1	0.8667	1.1062	0.8667	1.1062	0.8667	1.1062
0.2	0.8668	1.1063	0.8668	1.1064	0.8668	1.1064
0.4	0.8674	1.1088	0.8687	1.1124	0.8688	1.1125
0.6	0.8692	1.1198	0.8750	1.1405	0.8765	1.1436
0.8	0.8735	1.1686	0.8893	1.2445	0.8945	1.2584
0.975	0.8881	1.7064	0.9280	2.0652	0.9422	2.1280
1.0	0.9334	∞	1.0258	∞	1.0503	∞

Table 8 The normalized stress intensity factors for two unequal cracks in an infinite inhomogeneous orthotropic medium under uniform crack surface pressure, $\sigma_{yy}(x,0) = -p_0$, $E(x_1) = E_0\exp(\alpha x_1)$, $\nu = 0.3$, $(c_1+c_2)/d = 0.5$.

$\kappa = -0.25$, $c_2/c_1 = 0.25$					$\kappa = 0.0$, $c_2/c_1 = 0.25$				
αc_1	$\frac{k_1(a_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(a_2)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_2)}{p_0\sqrt{c_1}}$	αc_1	$\frac{k_1(a_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(a_2)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_2)}{p_0\sqrt{c_1}}$
0.0	1.00429	1.01010	0.55216	0.54146	0.0	1.00429	1.01010	0.55216	0.54146
0.01	1.00171	1.01253	0.55240	0.54232	0.01	1.00171	1.01252	0.55240	0.54231
0.10	0.97825	1.03400	0.55437	0.54991	0.10	0.97786	1.03357	0.55421	0.54976
0.25	0.93900	1.06872	0.55710	0.56227	0.25	0.93743	1.06671	0.55646	0.56166
0.50	0.87545	1.12420	0.56016	0.58189	0.50	0.87163	1.11809	0.55856	0.58038
0.75	0.81612	1.17694	0.56131	0.60011	0.75	0.81038	1.16556	0.55884	0.59774
1.00	0.76203	1.22710	0.56061	0.61679	1.00	0.75489	1.20973	0.55749	0.61377
1.25	0.71342	1.27482	0.55823	0.63194	1.25	0.70538	1.25104	0.55471	0.62849
1.50	0.67015	1.32028	0.55437	0.64562	1.50	0.66160	1.28987	0.55069	0.64197
1.75	0.63181	1.36367	0.54923	0.65794	1.75	0.62305	1.32655	0.54562	0.65430
2.00	0.59792	1.40519	0.54304	0.66901	2.00	0.58915	1.36137	0.53969	0.66557

Table 8 (Continued)

$\kappa = 0.5, \quad c_2/c_1 = 0.25$

αc_1	$\frac{k_1(a_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(a_2)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_2)}{p_0\sqrt{c_1}}$
0.0	1.00429	1.01010	0.55216	0.54146
0.01	1.00170	1.01251	0.55239	0.54231
0.10	0.97746	1.03312	0.55404	0.54960
0.25	0.93578	1.06459	0.55575	0.56099
0.50	0.86754	1.11153	0.55669	0.57857
0.75	0.80420	1.15325	0.55579	0.59474
1.00	0.74718	1.19085	0.55341	0.60968
1.25	0.69669	1.22509	0.54981	0.62349
1.50	0.65236	1.25657	0.54521	0.63626
1.75	0.61357	1.28579	0.53981	0.64809
2.00	0.57964	1.31314	0.53375	0.65906

$\kappa = 1.0, \quad c_2/c_1 = 0.25$

αc_1	$\frac{k_1(a_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(a_2)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_2)}{p_0\sqrt{c_1}}$
0.0	1.00429	1.01010	0.55216	0.54146
0.01	1.00170	1.01251	0.55239	0.54231
0.10	0.97725	1.03289	0.55395	0 .54952
0.25	0.93491	1.06348	0.55536	0.56061
0.50	0.86538	1.10805	0.55563	0.57753
0.75	0.80091	1.14668	0.55399	0.59294
1.00	0.74307	1.18072	0.55090	0.60712
1.25	0.69204	1.21113	0.54669	0.62021
1.50	0.64742	1.23862	0.54157	0.63234
1.75	0.60850	1.26377	0.53577	0.64362
2.00	0.57457	1.28702	0.52942	0.65412

$\kappa = 5.0, \quad c_2/c_1 = 0.25$

αc_1	$\frac{k_1(a_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(a_2)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_2)}{p_0\sqrt{c_1}}$
0.0	1.00429	1.01010	0.55216	0.54146
0.01	1.00169	1.01250	0.55239	0.54231
0.10	0.97681	1.03240	0.55376	0.54933
0.25	0.93307	1.06111	0.55451	0.55979
0.50	0.86075	1.10060	0.55317	0.57510
0.75	0.79385	1.13250	0.54967	0.58855
1.00	0.73421	1.15878	0.54467	0.60059
1.25	0.68203	1.18078	0.53860	0.61150
1.50	0.63676	1.19948	0.53179	0.62148
1.75	0.59757	1.21562	0.52447	0.63068
2.00	0.56361	1.22978	0.51680	0.63921

$\kappa = -0.25, \quad c_2/c_1 = 1.0$

αc_1	$\frac{k_1(a_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(a_2)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_2)}{p_0\sqrt{c_1}}$
0.0	1.02796	1.04795	1.04795	1.02796
0.01	1.02477	1.04978	1.0461	1.0311
0.10	.99602	1.06556	1.0288	1.0591
0.25	.94961	1.09128	0.99839	1.1045
0.50	.87927	1.13545	0.94573	1.1764
0.75	.81729	1.18161	0.89313	1.2432
1.00	.76231	1.22847	0.84244	1.3046
1.25	.71346	1.27474	0.79476	1.3608
1.50	.67013	1.31969	0.75064	1.4121
1.75	.63179	1.36299	0.71025	1.4592
2.00	.59791	1.40459	0.67350	1.5024

$\kappa = 0.0, \quad c_2/c_1 = 1.0$

αc_1	$\frac{k_1(a_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(a_2)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_2)}{p_0\sqrt{c_1}}$
0.0	1.02796	1.04795	1.04795	1.02796
0.01	1.02476	1.04977	1.0461	1.0311
0.10	0.99552	1.06498	1.0282	1.0585
0.25	0.94786	1.08894	0.99606	1.0190?
0.50	0.87537	1.12905	0.94040	1.1691
0.75	0.81153	1.17010	0.88560	1.2306
1.00	0.75517	1.21108	0.83370	1.2865
1.25	0.70543	1.25099	0.78567	1.3373
1.50	0.66159	1.28932	0.74185	1.3835
1.75	0.62304	1.32590	0.70217	1.4255
2.00	0.58914	1.36080	0.66641	1.4640

$\kappa = 0.5, \quad c_2/c_1 = 1.0$

αc_1	$\frac{k_1(a_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(a_2)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_2)}{p_0\sqrt{c_1}}$
0.0	1.02796	1.04795	1.04795	1.02796
0.01	1.02475	1.04975	1.0461	1.0311
0.10	0.99499	1.06435	1.0276	1.0579
0.25	0.94598	1.08641	0.99350	1.0990
0.50	0.87115	1.12210	0.93434	1.1608
0.75	0.80530	1.15758	0.87674	1.2159
1.00	0.74746	1.19212	0.82307	1.2651
1.25	0.69673	1.22504	0.77421	1.3092
1.50	0.65235	1.25606	0.73029	1.3489
1.75	0.61356	1.28518	0.69109	1.3848
2.00	0.57964	1.31259	0.65616	1.4175

Table 8 (Continued)

$\kappa = 1.0$, $c_2/c_1 = 1.0$					$\kappa = 5.0$, $c_2/c_1 = 1.0$				
αc_1	$\frac{k_1(a_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(a_2)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_2)}{p_0\sqrt{c_1}}$	αc_1	$\frac{k_1(a_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(a_2)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_2)}{p_0\sqrt{c_1}}$
0.0	1.02796	1.04795	1.04795	1.02796	0.0	1.02796	1.04795	1.04795	1.02796
0.01	1.02474	1.04975	1.0461	1.0311	0.01	1.02473	1.04973	1.0461	1.0311
0.10	0.99471	1.06402	1.0272	1.0575	0.10	0.99412	1.06333	1.0265	1.0568
0.25	0.94499	1.08506	0.99213	1.0974	0.25	0.94284	1.08218	0.98913	1.0940
0.50	0.86890	1.11838	0.93097	1.1562	0.50	0.86404	1.11032	0.92338	1.1457
0.75	0.80198	1.15086	0.87168	1.2075	0.75	0.79481	1.13628	0.85991	1.1881
1.00	0.74334	1.18194	0.81682	1.2527	1.00	0.73445	1.15979	0.80183	1.2234
1.25	0.69209	1.21107	0.76727	1.2927	1.25	0.68207	1.18065	0.75005	1.2530
1.50	0 .64741	1.23812	0.72308	1.3285	1.50	0.63675	1.19898	0.70449	1.2784
1.75	0.60849	1.26318	0.68390	1.3606	1.75	0.59756	1.21507	0.66462	1.3002
2.00	0.57456	1.28649	0.64923	1.3897	2.00	0.56360	1.22929	0.62976	1.3194

Table 9 The normalized stress intensity factors for two unequal cracks in an infinite inhomogeneous orthotropic medium under uniform crack surface pressure, $\sigma_{yy}(x,0) = -p_0$, $E(x_1) = E_0\exp(\alpha x_1)$, $\nu = 0.3, \kappa = 0.5$.

$c_2/c_1 = 0.25$, $(c_1+c_2)/d = 0.75$				
αc_1	$\frac{k_1(a_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(a_2)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_2)}{p_0\sqrt{c_1}}$
0.0	1.01146	1.04850	0.66867	0.60894
0.01	1.00880	1.05090	0.66935	0.61018
0.10	0.98396	1.07123	0.67479	0.62088
0.25	0.94123	1.10193	0.68212	0.63757
0.50	0.87140	1.14689	0.69093	0.66332
0.75	0.80678	1.18608	0.69652	0.68704
1.00	0.74884	1.22084	0.69956	0.70907
1.25	0.69772	1.25217	0.70055	0.72964
1.50	0.65298	1.28079	0.69990	0.74892
1.75	0.61394	1.30729	0.69792	0.76707
2.00	0.57986	1.33211	0.69487	0.78420

$c_2/c_1 = 0.75$, $(c_1+c_2)/d = 0.75$				
αc_1	$\frac{k_1(a_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(a_2)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_2)}{p_0\sqrt{c_1}}$
0.0	1.04964	1.14076	1.04573	0.94551
0.01	1.04636	1.14252	1.0452	0.94823
0.10	1.01574	1.15652	1.0389	0.97155
0.25	0.96426	1.17566	1.0244	1.0077
0.50	0.88389	1.20191	0.99440	1.0632
0.75	0.81314	1.22513	0.96126	1.1138
1.00	0.75192	1.24726	0.92738	1.1604
1.25	0.69914	1.26906	0.89413	1.2034
1.50	0.65361	1.29073	0.86230	1.2432
1.75	0.61421	1.31230	0.83227	1.2802
2.00	0.57996	1.33373	0.80422	1.3146

$c_2/c_1 = 1.0$, $(c_1+c_2)/d = 0.75$				
αc_1	$\frac{k_1(a_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(a_2)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_2)}{p_0\sqrt{c_1}}$
0.0	1.06844	1.17318	1.17318	1.06844
0.01	1.06475	1.17451	1.1717	1.0720
0.10	1.03045	1.18417	1.1565	1.1028
0.25	0.97370	1.19589	1.1262	1.1500
0.50	0.88781	1.21176	1.0707	1.2213
0.75	0.81450	1.22790	1.0150	1.2853
1.00	0.75226	1.24581	0.96240	1.3434
1.25	0.69915	1.26541	0.91427	1.3963
1.50	0.65354	1.28619	0.87088	1.4448
1.75	0.61414	1.30762	0.83206	1.4894
2.00	0.57991	1.32931	0.79742	1.5307

Table 9 (Continued)

$c_2/c_1 = 0.25, \quad \alpha c_1 = 0.5$					$c_2/c_1 = 0.75, \quad \alpha c_1 = 0.5$				
$\frac{c_1+c_2}{d}$	$\frac{k_1(a_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(a_2)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_2)}{p_0\sqrt{c_1}}$	$\frac{c_1+c_2}{d}$	$\frac{k_1(a_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(a_2)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_2)}{p_0\sqrt{c_1}}$
0.0	0.8661	1.1053	0.4331	0.5527	0.0	0.8661	1.1053	0.7501	0.9572
0.1	0.86614	1.10527	0.48805	0.51930	0.1	0.86614	1.10526	0.78301	0.94138
0.2	0.86619	1.10539	0.49731	0.52839	0.2	0.86622	1.10545	0.79236	0.95090
0.3	0.86640	1.10607	0.51092	0.54096	0.3	0.86674	1.10694	0.80597	0.96313
0.4	0.86683	1.10783	0.53004	0.55743	0.4	0.86809	1.11143	0.82517	0.97818
0.5	0.86754	1.11153	0.55669	0.57857	0.5	0.87055	1.12107	0.85224	0.99643
0.6	0.86863	1.11885	0.59451	0.60569	0.6	0.87439	1.13943	0.89137	1.0187
0.7	0.87026	1.13374	0.65093	0.64115	0.7	0.88008	1.17380	0.95114	1.0464
0.8	0.87287	1.16752	0.74431	0.68972	0.8	0.88862	1.24313	1.0531	1.0828
0.9	0.87776	1.26963	0.94244	0.76440	0.9	0.90295	1.42220	1.2792	1.1376
0.95	0.88273	1.44262	1.1969	0.82657	0.95	0.91588	1.68871	1.5825	1.1836
1.0	0.9326	∞	∞	1.2606	1.0	1.0244	∞	∞	1.5458

$c_2/c_1 = 1.0, \quad \alpha c_1 = 0.5$			
$\frac{k_1(a_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_1)}{p_0\sqrt{c_1}}$	$\frac{k_1(a_2)}{p_0\sqrt{c_1}}$	$\frac{k_1(b_2)}{p_0\sqrt{c_1}}$
0.8661	1.1053	0.8661	1.1053
0.86614	1.10526	0.86986	1.1096
0.86620	1.10541	0.87854	1.1187
0.86671	1.10682	0.89116	1.1302
0.86823	1.11153	0.90900	1.1441
0.87115	1.12210	0.93434	1.1608
0.87588	1.14252	0.97136	1.1810
0.88301	1.18072	1.0287	1.2061
0.89377	1.25695	1.1282	1.2392
0.91169	1.45009	1.3533	1.2894
0.92762	1.73266	1.6602	1.3323
1.0561	∞	∞	1.6792

Table 10 Stress intensity factors and the contact zone sizes for an isotropic inhomogeneous medium under fixed grip remote bending, $\nu = 0.3$, $k_0 = \widehat{E}_0\varepsilon_1\sqrt{a}$, $\sigma_{yy}(x,0) = -\widehat{E}_0\varepsilon_1(x/a)\exp(\alpha x)$, $\widehat{E}_0 = E_0/(1-\nu^2)$.

	$\varepsilon_1 > 0$		$\varepsilon_1 < 0$		No contact	
αa	b/a	$\frac{k_1(a)}{\widehat{E}_0\varepsilon_1\sqrt{a}}$	b/a	$\frac{k_1(-a)}{-\widehat{E}_0\varepsilon_1\sqrt{a}}$	$\frac{k_1(a)}{\widehat{E}_0\varepsilon_1\sqrt{a}}$	$\frac{k_1(-a)}{\widehat{E}_0\varepsilon_1\sqrt{a}}$
0.0	0.3333	0.5443	0.3333	0.5443	0.5	−0.5
0.1	0.3405	0.5964	0.3258	0.4965	0.5523	−0.4523
0.25	0.3508	0.6832	0.3140	0.4320	0.6402	−0.3886
0.5	0.3669	0.8554	0.2947	0.3419	0.8151	−0.3010
1.0	0.3968	1.3359	0.2585	0.2130	1.3029	−0.1794
1.5	0.4234	2.0827	0.2269	0.1320	2.0570	−0.1064
2.0	0.4476	3.2488	0.2000	0.0815	3.2297	−0.0631

Table 11 Stress intensity factors and contact zone sizes for an isotropic inhomogeneous medium under remote strain $\varepsilon_{yy} = \varepsilon_0 + \varepsilon_1(x/a) = \varepsilon_1(\beta + x/a)$, (Fig. 25), $\alpha = 0.5$, $\nu = 0.3$, $\varepsilon_0/\varepsilon_1 = \beta$, $\varepsilon_1 > 0$.

$\beta = \frac{\varepsilon_0}{\varepsilon_1}$	b/a	$\frac{k_1(a)}{\widehat{E}_0\varepsilon_1\sqrt{a}}$	$\frac{k_1(-a)}{\widehat{E}_0\varepsilon_1\sqrt{a}}$
0.0	0.3669	0.8554	0.0
0.25	0.7184	1.1798	0.0
0.4485	1	1.4561	0.0
0.45	1	1.4582	0.0010
0.50	1	1.5297	0.0345
0.625	1	1.7083	0.1184
0.75	1	1.8869	0.2023
1.00	1	2.2442	0.3700